Homework for week 6 (PDF)

1. **Derivative of Fermi-Dirac function** (K&K 6.1) Show that $-\frac{\partial f}{\partial \varepsilon}$ evaluated at the Fermi level $\varepsilon = \mu$ has the value $\frac{1}{4\pi kT}$. Thus the lower the temperature, the steeper the slope of the Fermi-Dirac function.

2. **Symmetry of filled and vacant orbitals** (K&K 6.2) Show that

\[ f(\mu + \delta) = 1 - f(\mu - \delta) \] (1)

Thus the probability that an orbital $\delta$ above the Fermi level is occupied is equal to the probability an orbital $\delta$ below the Fermi level is vacant. A vacant orbital is sometimes known as a hole.

3. **Distribution function for double occupancy statistics** (K&K 6.3) Let us imagine a new mechanics in which the allowed occupancies of an orbital are 0, 1, and 2. The values of the energy associated with these occupancies are assumed to be 0, $\varepsilon$, and $2\varepsilon$, respectively.

   a) Derive an expression for the ensemble average occupancy $\langle N \rangle$, when the system composed of this orbital is in thermal and diffusive contact with a reservoir at temperature $T$ and chemical potential $\mu$.
   
   b) Return now to the usual quantum mechanics, and derive an expression for the ensemble average occupancy of an energy level which is doubly degenerate; that is, two orbitals have the identical energy $\varepsilon$. If both orbitals are occupied the total energy is $2\varepsilon$. How does this differ from part (a)?

4. **Entropy of mixing** (Modified from K&K 6.6) Suppose that a system of $N$ atoms of type $A$ is placed in diffusive contact with a system of $N$ atoms of type $B$ at the same temperature and volume.

   a) Show that after diffusive equilibrium is reached the total entropy is increased by $2Nk \ln 2$. The entropy increase $2Nk \ln 2$ is known as the entropy of mixing.
   
   b) If the atoms are identical ($A = B$), show that there is no increase in entropy when diffusive contact is established. The difference has been called the Gibbs paradox.
   
   c) Since the Helmholtz free energy is lower for the mixed $AB$ than for the separated $A$ and $B$, it should be possible to extract work from the mixing process. Construct a process that could extract work as the two gases are mixed at fixed temperature. You will probably need to use walls that are permeable to one gas but not the other.

   **Note** This course has not yet covered work, but it was covered in Energy and Entropy, so you may need to stretch your memory to finish part (c).

5. **Ideal gas in two dimensions** (K&K 6.12)

   a) Find the chemical potential of an ideal monatomic gas in two dimensions, with $N$ atoms confined to a square of area $A = L^2$. The spin is zero.
   
   b) Find an expression for the energy $U$ of the gas.
   
   c) Find an expression for the entropy $S$. The temperature is $kT$.

6. **Ideal gas calculations** (K&K 6.14) Consider one mole of an ideal monatomic gas at 300K and 1 atm. First, let the gas expand isothermally and reversibly to twice the initial volume; second, let this be followed by an isentropic expansion from twice to four times the original volume.

   a) How much heat (in joules) is added to the gas in each of these two processes?
   
   b) What is the temperature at the end of the second process?
   
   c) Suppose the first process is replaced by an irreversible expansion into a vacuum, to a total volume twice the initial volume. What is the increase of entropy in the irreversible expansion, in J/K?