Homework for week 4 (PDF)

1. **Radiation in an empty box** As discussed in class, we can consider a black body as a large box with a small hole in it. If we treat the large box a metal cube with side length \( L \) and metal walls, the frequency of each normal mode will be given by:

\[
\omega_{n_x n_y n_z} = \frac{\pi c}{L} \sqrt{n_x^2 + n_y^2 + n_z^2}
\]

(1)

where each of \( n_x, n_y, \) and \( n_z \) will have positive integer values. This simply comes from the fact that a half wavelength must fit in the box. There is an additional quantum number for polarization, which has two possible values, but does not affect the frequency. Each normal mode is a harmonic oscillator, with energy eigenstates \( E_n = n \hbar \omega \) where we will not include the zero-point energy \( \frac{1}{2} \hbar \omega \), since that energy cannot be extracted from the box. (See the Casimir effect for an example where the zero point energy of photon modes does have an effect.)

**Note** This is a slight approximation, as the boundary conditions for light are a bit more complicated. However, for large \( n \) values this gives the correct result.

a) Show that the free energy is given by

\[
F = 8\pi \frac{V(kT)^4}{\hbar^3 c^3} \int_0^\infty \ln \left(1 - e^{-\xi}\right) \xi^2 d\xi
\]

(2)

\[
= -\frac{8\pi^5}{45} \frac{V(kT)^4}{\hbar^3 c^3}
\]

(3)

\[
= -\frac{\pi^2}{45} \frac{V(kT)^4}{\hbar^3 c^3}
\]

(4)

provided the box is big enough that \( \frac{\hbar c}{kT} \ll 1 \). Note that you may end up with a slightly different dimensionless integral that numerically evaluates to the same result, which would be fine. I also do not expect you to solve this definite integral analytically, a numerical confirmation is fine. However, you must manipulate your integral until it is dimensionless and has all the dimensionful quantities removed from it!

b) Show that the entropy of this box full of photons at temperature \( T \) is

\[
S = \frac{32\pi^5}{45} kV \left(\frac{kT}{\hbar c}\right)^3
\]

(5)

\[
= \frac{4\pi^2}{45} kV \left(\frac{kT}{\hbar c}\right)^3
\]

(6)

c) Show that the internal energy of this box full of photons at temperature \( T \) is

\[
U = \frac{8\pi^5}{15} \frac{(kT)^4}{\hbar^3 c^3}
\]

(7)

\[
= \frac{\pi^2}{15} \frac{(kT)^4}{\hbar^3 c^3}
\]

(8)

2. **Surface temperature of the earth** (K&K 4.5)

Calculate the temperature of the surface of the Earth on the assumption that as a black body in thermal equilibrium it reradiates as much thermal radiation as it receives from the Sun. Assume also that the surface of the Earth is a constant temperature over the day-night cycle. Use the sun’s surface temperature \( T_\odot = 5800 \) K; and the sun’s radius \( R_\odot = 7 \times 10^{10} \text{cm} \); and the Earth-Sun distance of \( 1.5 \times 10^{13} \text{cm} \).

3. **Pressure of thermal radiation** (modified from K&K 4.6) We discussed in class that

\[
p = -\left(\frac{\partial F}{\partial V}\right)_T
\]

(9)

Use this relationship to show that

a)

\[
p = -\sum_j \langle n_j \rangle \hbar \left(\frac{d\omega_j}{dV}\right)_T
\]

(10)
where $\langle n_j \rangle$ is the number of photons in the mode $j$:

b) Solve for the relationship between pressure and internal energy.

4. **Heat shields** (K&K 4.8) A black (nonreflective) plane at high temperature $T_h$ is parallel to a cold black plane at temperature $T_c$. The net energy flux density in vacuum between the two planes is $J_U = \sigma_B (T_h^4 - T_c^4)$, where $\sigma_B$ is the Stefan-Boltzmann constant used in (26). A third black plane is inserted between the other two and is allowed to come to a steady state temperature $T_m$. Find $T_m$ in terms of $T_h$ and $T_c$, and show that the net energy flux density is cut in half because of the presence of this plane. This is the principle of the heat shield and is widely used to reduce radiant heat transfer. *Comment:* The result for $N$ independent heat shields floating in temperature between the planes $T_u$ and $T_l$ is that the net energy flux density is $J_U = \sigma_B \frac{T_u^4 - T_l^4}{N+1}$.

5. **Heat capacity of vacuum**

a) Solve for the heat capacity of a vacuum, given the above, and assuming that photons represent all the energy present in vacuum.

b) Compare the heat capacity of vacuum at room temperature with the heat capacity of an equal volume of water.