Symmetries and Idealizations Homework 6

Due Friday 10/16

Problem 6.1 Square loop (practice) (based on Griffiths 2.4) Consider a square loop with each side a length a carrying a uniform linear charge density λ.

a) Find the electric field at the center of the square.

Solution:

The electric field to to each of the opposite sides will be in an opposite direction, so the total electric field at the center of the square is zero.

b) Find the work needed to bring a charge in from infinity to the center of the square. (You may start with the electrostatic potential due to a single finite line of charge.)

Solution:

The work needed to bring a charge Q in from infinity to the origin is just the charge times the electrostatic potential at the origin.

$$U = QV(0) \tag{11.25}$$

The electrostatic potential due to a finite line charge, in the plane that bisects the line, is

$$V(r) = k\lambda \ln\left(\frac{\sqrt{a^2+4r^2}+a}{\sqrt{a^2+4r^2}-a}\right) \tag{11.26}$$

$$V(l/2) = k\lambda \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \tag{11.27}$$

Since there are four sides, we just end up with

$$U = 4kQ\lambda \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \tag{11.28}$$

Problem 6.2 Three charges (practice) (Griffiths 2.32) Three charges are situated at the corners of a square (side s). Two have charge $-q$ and are located on opposite corners. The third has charge $+q$ and is opposite an empty corner.

a) How much work does it take to bring in another charge, $+q$, from far away and place it at the fourth corner?

Solution:

To find the work needed, I need to calculate the potential at the final location of the fourth charge due to the presence of the other charges. (The subscripts denote the charge that is creating the potential.)

$$\begin{aligned}
W &= q\,V(\vec{\boldsymbol{r}}) \\
&= q\,(V_1 + V_2 + V_3) \\
&= q\left(\frac{k(-q)}{s} + \frac{k(q)}{\sqrt{2}s} + \frac{k(-q)}{s}\right) \\
&= \frac{-k\,q^2}{s}\left(2 - \frac{\sqrt{2}}{2}\right)
\end{aligned}$$

The negative sign indicates that the charge is losing energy is being placed at the corner of the square.

b) How much work does it take to assemble the whole configuration of four charges?

Solution:

$$W = \frac{1}{2}\sum_{i=1}^{4} q_i V(\vec{\boldsymbol{r}}_i)$$

The potential for charges on opposite corners is the same (here, the subscripts denote the total potential at the location of the charge due to the presence of the other four charges):

$$V_{T1} = V_{T3} = \frac{k\,q^2}{s}\left(2 - \frac{\sqrt{2}}{2}\right)$$

$$V_{T2} = V_{T4} = \frac{-k\,q^2}{s}\left(2 - \frac{\sqrt{2}}{2}\right)$$

$$\begin{aligned}
W &= \frac{1}{2}\left[2(-q)V_1 + 2(q)V_2\right] \\
&= \frac{(-q)\,k\,q^2}{s}\left(2 - \frac{\sqrt{2}}{2}\right) + \frac{-(q)\,k\,q^2}{s}\left(2 - \frac{\sqrt{2}}{2}\right)
\end{aligned}$$

$$= \quad -2\,\frac{k\,q^2}{s}\left(2-\frac{\sqrt{2}}{2}\right)$$

The negative sign indicates that the system loses some energy in order to be placed in the square.

Problem 6.3 Problem 5.4 (c) and (d) Remember to turn in parts c and d of problem 5.4 from the previous homework, if you didn't turn them in on Wednesday!

Problem 6.4 A thin spherical shell (challenge) Consider a thin spherical shell with radius R and total charge Q.

a) What is the surface charge density of the shell?

b) Find the electrostatic potential everywhere within the shell.

c) Compute the gradient of the electrostatic potential to find the electric field *within* the shell.

d) Find the electrostatic potential everywhere outside the shell.

e) Compute the gradient of the electrostatic potential to find the electric field *outside* the shell.

f) Comment on your results.

Solution:

a) The charge over the area is

$$\sigma = \frac{Q}{4\pi R^2}$$

b) The potential is clearly spherically symmetric by symmetry, so I only need to work out $V(r)$. We can get this by just adding up the potential due to all the little bits of charge.

$$V(r) = \int \frac{kdQ'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \tag{11.29}$$

In this case, we're talking about a surface, so $dQ' = \sigma dA'$

$$V(r) = k\sigma \int \frac{dA'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \tag{11.30}$$

The obvious set of coordinates is spherical, and since the only thing that isn't spherically symmetric is the point $\vec{\boldsymbol{r}}$ we're curious about, let's choose our coordinate system so that that point is on the z axis, to maintain as much symmetry as we can.

$$\begin{aligned}
V(r) &= k\sigma \iint \frac{R^2 d\varphi \sin\theta d\theta}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} && (11.31)\\
&= k\sigma R^2 \iint \frac{1}{\sqrt{(R\cos\theta - r)^2 + R^2\sin^2\theta}} d\varphi \sin\theta d\theta && (11.32)\\
&= 2\pi k\sigma R^2 \int_0^\pi \frac{1}{\sqrt{(R\cos\theta - r)^2 + R^2\sin^2\theta}} \sin\theta d\theta && (11.33)\\
&= 2\pi k\sigma R^2 \int_0^\pi \frac{1}{\sqrt{R^2\cos^2\theta + r^2 - 2R\cos\theta r + R^2\sin^2\theta}} \sin\theta d\theta && (11.34)\\
&= 2\pi k\sigma R^2 \int_0^\pi \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} \sin\theta d\theta && (11.35)\\
&= 2\pi k\sigma R^2 \int_{-1}^1 \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} d\cos\theta && (11.36)
\end{aligned}$$

Here we just define

$$\xi = R^2 + r^2 - 2Rr\cos\theta \qquad\qquad d\xi = -2Rr d\cos\theta \tag{11.37}$$

$$\begin{aligned}
V(r) &= 2\pi k\sigma R^2 \int_{R^2+r^2+2Rr}^{R^2+r^2-2Rr} \frac{1}{\sqrt{\xi}} \left(-\frac{1}{2Rr} d\xi\right) && (11.38)\\
&= \frac{k\pi\sigma R}{r} \int_{R^2+r^2-2Rr}^{R^2+r^2+2Rr} \frac{1}{\sqrt{\xi}} d\xi && (11.39)\\
&= \frac{k\pi\sigma R}{r} \int_{R^2+r^2+-Rr}^{R^2+r^2+2Rr} \frac{1}{\sqrt{\xi}} d\xi && (11.40)\\
&= \frac{k\pi\sigma R}{r}\, 2\sqrt{\xi}\Big|_{R^2+r^2-2Rr}^{R^2+r^2+2Rr} && (11.41)\\
&= \frac{k\pi\sigma R}{r} 2\left(\sqrt{R^2 + r^2 - 2Rr} - \sqrt{R^2 + r^2 + 2Rr}\right) && (11.42)\\
&= \frac{k\pi\sigma R}{r} 2\left(\sqrt{(R+r)^2} - \sqrt{(R-r)^2}\right) && (11.43)\\
&= \frac{k\pi\sigma R}{r} 2\left(R + r - (R - r)\right) && (11.44)\\
&= \frac{k\pi\sigma R}{r} 4r && (11.45)\\
&= k4\pi\sigma R && (11.46)\\
&= \frac{kQ}{R} && (11.47)
\end{aligned}$$

Note that we assumed that we are talking about the inside of the shell, when converting $\sqrt{(R-r)^2}$ to $R-r$.

c) The electric field is the gradient of the potential. Since the potential is independent of $\vec{\boldsymbol{r}}$ (it involve r, θ or φ), its gradient is zero and the electric field is zero inside the shell.

d) The outside is found in a manner precisely analogous to the inside.

$$
\begin{aligned}
V(r) &= 2\pi k\sigma R^2 \int_{R^2+r^2+2Rr}^{R^2+r^2-2Rr} \frac{1}{\sqrt{\xi}}\left(-\frac{1}{2Rr}d\xi\right) && (11.48)\\
&= \frac{k\pi\sigma R}{r}\int_{R^2+r^2-2Rr}^{R^2+r^2+2Rr}\frac{1}{\sqrt{\xi}}d\xi && (11.49)\\
&= \frac{k\pi\sigma R}{r}\int_{R^2+r^2-2Rr}^{R^2+r^2+2Rr}\frac{1}{\sqrt{\xi}}d\xi && (11.50)\\
&= \frac{k\pi\sigma R}{r}\,2\sqrt{\xi}\Big|_{R^2+r^2-2Rr}^{R^2+r^2+2Rr} && (11.51)\\
&= \frac{k\pi\sigma R}{r}2\left(\sqrt{R^2+r^2+2Rr}-\sqrt{R^2+r^2-2Rr}\right) && (11.52)\\
&= \frac{k\pi\sigma R}{r}2\left(\sqrt{(R+r)^2}-\sqrt{(R-r)^2}\right) && (11.53)\\
&= \frac{k\pi\sigma R}{r}2\left(r+R-(r-R)\right) && (11.54)\\
&= \frac{k\pi\sigma R}{r}4R && (11.55)\\
&= k4\pi\sigma R && (11.56)\\
&= \frac{kQ}{r} && (11.57)
\end{aligned}
$$

e) To find the electric field outside the shell, we just need to compute the gradient of $\frac{kQ}{r}$. This is

$$
\begin{aligned}
\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) &= -\vec{\nabla}\frac{kQ}{r} && (11.58)\\
\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) &= -\left(\frac{\partial}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\phi}\right)\frac{kQ}{r} && (11.59)\\
&= -\hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{kQ}{r} && (11.60)\\
&= \hat{\mathbf{r}}\frac{kQ}{r^2} && (11.61)
\end{aligned}
$$

f) The two answers look quite similar. However, there is a *huge* difference between R and r. The former is the radius of the shell, while the latter is the distance between the center of the shell and the position whose electrostatic potential we want to know.

The answer outside the shell

$$\frac{kQ}{r}$$

is the same as the electrostatic potential due to a point charge! So *outside* a spherical shell of charge, it behaves as if it were simply a point charge. In contrast, the potential inside the shell

$$\frac{kQ}{R}$$

is completely independent of position, which means that there is no work required to move a point charge around within the charged shell, and there is no electric field inside the shell. These are interesting and deep results.