Symmetries and Idealizations Homework 5

Due Wednesday 10/14

Problem 5.1 Directional derivative (practice) Imagine you're standing on a landscape with a local topology described by the function $f = kx^2y$, where $k = 20\frac{m}{km^3}$ is a constant, x and y are east and north coordinates, respectively, with units of kilometers. You're standing at the spot (3 km,2 km) and there is a cottage located at (1 km, 2 km). At the spot you're standing, what is the slope of the ground in the direction of the cottage? Plot the function f in Maple. Does your result makes sense with the picture?

Solution to problem 5.1 Directional derivative The slope is a directional derivative. First find the gradient of the function f at your location:

$$\begin{aligned}\vec{\nabla} f &= (\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y})\,f && (11.25)\\ &= (2kxy\,\hat{\imath} + kx^2\,\hat{\jmath}) && (11.26)\\ &= (240\,\hat{\imath} + 180\,\hat{\jmath})\frac{m}{km} && (11.27)\end{aligned}$$

The gradient contains all of the information needed to find the slope in any direction. Notice that, since f is a function of only two variables, the gradient is a 2-d vector, i.e. it lives on the topo map, not in three-dimensional space.

Next, find the 2-d unit vector that points in the direction pointing from your current position to the cottage.

$$\begin{aligned}\vec{r}_{\text{you}} &= 3\hat{\imath} + 2\hat{\jmath} && (11.28)\\ \vec{r}_{\text{cottage}} &= 1\hat{\imath} + 2\hat{\jmath} && (11.29)\\ \vec{r} &= \vec{r}_{\text{cottage}} - \vec{r}_{\text{you}} = -2\hat{\imath} && (11.30)\\ \hat{r} &= \frac{\vec{r}}{|\vec{r}|} = -\hat{\imath} && (11.31)\end{aligned}$$

The directional derivative that you want is the dot product of the gradient with the appropriate unit vector:

$$\begin{aligned}\vec{\nabla} f \cdot \hat{r} &= (240\,\hat{\imath} + 180\,\hat{\jmath})\frac{m}{km} \cdot -\hat{\imath} && (11.32)\\ &= -240\frac{m}{km} && (11.33)\end{aligned}$$

Problem 5.2 Infinite disk Find the electrostatic potential due to an infinite disk, using your results from the finite disk problem.

Solution to problem 5.2 Infinite disk The potential appears to become infinite as $R \to \infty$. This is not physical, and is a typical problem when dealing with charge distributions which go off to infinity. Don't forget that we assumed that the potential was zero at infinity, but this will fail if there is charge there! The resolution to this problem is to place the zero of potential somewhere else, where there is no charge. For instance, if we decide that this should be along the axis at $z = z_0$, then what we are really interested in is the *difference* between the potential at z and at z_0. We must therefore *renormalize* our previous answer by subtracting the potential at z_0, resulting in

$$V(z) = \frac{\sigma}{4\pi\epsilon_0}\left[\left(2\pi\sqrt{R^2+z^2} - 2\pi z\right) - \left(2\pi\sqrt{R^2+z_0^2} - 2\pi z_0\right)\right]$$

If $R \gg z$, then

$$\begin{aligned}\sqrt{R^2+z^2} &= R\sqrt{1+\frac{z^2}{R^2}} \\ &= R\left(1+\frac{1}{2}\frac{z^2}{R^2} - \frac{1}{8}\frac{z^4}{R^4} + \ldots\right)\end{aligned}$$

Inserting this into the renormalized result above, the two terms in R cancel, all terms in with R in the denominator go to 0, and we are left with

$$V(z) = \frac{\sigma}{2\epsilon_0}(z_0 - z)$$

Problem 5.3 A point charge

a) Give an expression for the electric potential $V(\vec{\boldsymbol{r}})$ at a point $\vec{\boldsymbol{r}}$ due to a point charge located at $\vec{\boldsymbol{r}}'$.

Solution:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

b) Give an expression for the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ at a point $\vec{\boldsymbol{r}}$ due to a point charge located at $\vec{\boldsymbol{r}}'$.

Solution:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{Q(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

c) Working in your favorite coordinate system, compute the gradient of V.

Solution:

In rectangular coordinates,

$$\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}' = (x-x')\,\hat{\imath} + (y-y')\,\hat{\jmath} + (z-z')\,\hat{k}$$

Thus,

$$\begin{aligned}
\vec{\nabla}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) &= \vec{\nabla}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)\\
&= \frac{\partial}{\partial x}\left(\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-1/2}\right)\hat{\imath} + ...\\
&= -\frac{1}{2}\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-3/2}\frac{\partial}{\partial x}\left((x-x')^2\right)\hat{\imath} + ...\\
&= -\frac{(x-x')\,\hat{\imath} + (y-y')\,\hat{\jmath} + (z-z')\,\hat{k}}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}}\\
&= -\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}
\end{aligned}$$

which shows that $\vec{\nabla}V = -\vec{\boldsymbol{E}}$, as expected. (This computation can be simplified using the chain rule, since

$$\vec{\nabla}\left(h^{-1/2}\right) = -\frac{1}{2}\,h^{-3/2}\,\vec{\nabla}h$$

Here $h = (x-x')^2+(y-y')^2+(z-z')^2$, so

$$\begin{aligned}
\vec{\nabla}h &= 2(x-x')\,\hat{\imath} + 2(y-y')\,\hat{\jmath} + 2(z-z')\,\hat{k}\\
&= 2(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')
\end{aligned}$$

which leads to the same result.)

d) Write several sentences comparing your answers to the last two questions.

Solution:

The key point is that $\vec{\boldsymbol{E}} = -\vec{\nabla}V$.

Problem 5.4 Line sources

a) Find the electric field around an infinite, uniformly charged, straight wire, starting from the expression for the electrostatic potential that we found in class:

$$V(\vec{r}) = \frac{2\lambda}{4\pi\epsilon_0}\,\ln\frac{r_0}{r}$$

Solution:

From the expression for V, we can find the electric field:

$$\begin{aligned}
\vec{E} &= -\vec{\nabla} V \\
&= -\frac{2\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial r}\frac{r_0}{r}\,\hat{r} \\
&= -\frac{2\lambda}{4\pi\epsilon_0}\frac{r}{r_0}\left(-\frac{r_0}{r^2}\right)\hat{r} \\
&= \frac{2\lambda}{4\pi\epsilon_0}\frac{\hat{r}}{r}
\end{aligned}$$

Notice that the dependence on the zero of potential r_0 has disappeared, as expected.

b) Find the electric field around a finite, uniformly charged, straight wire, at a point a distance r straight out from the midpoint, starting from the expression for the electrostatic potential that we found in class:

$$V(\vec{r}) = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(L+\sqrt{L^2+r^2}\right) - \ln\left(-L+\sqrt{L^2+r^2}\right)\right]$$

Solution:

From the expression for V, we can find the electric field:

$$\begin{aligned}
\vec{E} &= -\vec{\nabla} V \\
&= -\frac{\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left[\ln\left(L+\sqrt{L^2+r^2}\right) - \ln\left(-L+\sqrt{L^2+r^2}\right)\right]\hat{r} \\
&= -\frac{\lambda}{4\pi\epsilon_0}\left[\left(\frac{1}{L+\sqrt{L^2+r^2}}\right)\left(\frac{r}{\sqrt{L^2+r^2}}\right)\right. \\
&\qquad\qquad \left. -\left(\frac{1}{-L+\sqrt{L^2+r^2}}\right)\left(\frac{r}{\sqrt{L^2+r^2}}\right)\right]\hat{r} \\
&= -\frac{\lambda}{4\pi\epsilon_0}\left(\frac{r}{\sqrt{L^2+r^2}}\right)\left[\left(\frac{1}{L+\sqrt{L^2+r^2}}\right) - \left(\frac{1}{-L+\sqrt{L^2+r^2}}\right)\right]\hat{r} \\
&= -\frac{\lambda}{4\pi\epsilon_0}\left(\frac{r}{\sqrt{L^2+r^2}}\right)\left[\frac{\left(-L+\sqrt{L^2+r^2}\right) - \left(L+\sqrt{L^2+r^2}\right)}{-L^2+L^2+r^2}\right]\hat{r} \\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{2L}{r\sqrt{L^2+r^2}}\right)\hat{r}
\end{aligned}$$

c) Find the electric field around an infinite, uniformly charged, straight wire, starting from Coulomb's Law.

Solution:

Setting up Coulomb's Law for an infinite wire, I'll solve for a finite wire (length $2L$) and then take the limit that the length goes to infinity. The line segment lies along the z-axis, and I'm finding the electric field some distance r away from the center of the line segment. I'll use cylindrical coordinates to compute the electric field.

The vector pointing toward the point where the potential is evaluated is:

$$\vec{\boldsymbol{r}} = r\,\hat{r} = r\cos\phi\,\hat{\imath} + r\sin\phi\,\hat{\jmath}$$

(I actually don't need to rewrite $\vec{\boldsymbol{r}}$ with Cartesian basis vectors, because $\hat{r}$ doesn't change direction when I integrate, but I'll do it anyway.) The vector pointing toward an infinitesimal piece of charge is:

$$\vec{\boldsymbol{r}}' = z'\,\hat{z}$$

The electric field is:

$$\begin{aligned}
\vec{E} &= \lim_{L\to\infty}\int_{-L}^{L}\frac{k\lambda\,(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')\,ds'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} \\
&= \lim_{L\to\infty}\int_{-L}^{L}\frac{k\lambda\,(r\cos\phi\,\hat{\imath} + r\sin\phi\,\hat{\jmath} - z'\hat{k})\,dz'}{(r^2+z^2)^{3/2}}
\end{aligned}$$

I can break the integral into two pieces that can be integrated separately. The first piece becomes:

$$\begin{aligned}
\vec{E_r} &= \lim_{L\to\infty}\int_{-L}^{L}\frac{k\lambda\,(r\cos\phi\,\hat{\imath} + r\sin\phi\,\hat{\jmath})\,dz'}{(r^2+z^2)^{3/2}} \\
&= \lim_{L\to\infty}\left[\left.\frac{(r\cos\phi\,\hat{\imath} + r\sin\phi\,\hat{\jmath})\;z'}{r^2\;(r^2+z^2)^{1/2}}\right|_{-L}^{L}\right] \\
&= \lim_{L\to\infty}\frac{2Lk\lambda\,(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})}{r\;(r^2+L^2)^{1/2}}
\end{aligned}$$

There are a couple of options for evaluating this limit. One is to expand it in a power series of r^2/L^2:

$$\vec{E_r} = \lim_{L\to\infty}\frac{2Lk\lambda\,(\cos\phi\,\hat{\imath} + r\sin\phi\,\hat{\jmath})}{r}\;\frac{1}{L}\;\left(1 - \frac{1}{2}\,\frac{r}{L^2} + \ldots\right)$$

The L's cancel, and taking the limit, all the terms in the series disappear except the first term:

$$\begin{aligned}
\vec{E}_r &= \frac{2k\lambda\,(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})}{r} \\
\vec{E}_r &= \frac{2k\lambda}{r}\,\hat{r}
\end{aligned}$$

The second integral can be done by doing a u substitution:

$$\vec{E}_z = \lim_{L\to\infty} \int_{-L}^{L} \frac{-k\lambda\,\hat{k}\,dz'}{(r^2+z^2)^{3/2}}$$

Let $u = r^2 + z^2$, then $du = 2z\,dz$, and my integral becomes:

$$\begin{aligned}
\vec{E}_z &= \lim_{L\to\infty} \frac{k\lambda}{2} \int_{r^2+L^2}^{r^2+L^2} \frac{-du}{u^{3/2}}\hat{k} \\
&= \lim_{L\to\infty} k\lambda\,\hat{k}\;2\;\, u^{-1/2}\Big|_{r^2+L^2}^{r^2+L^2} \\
&= 0
\end{aligned}$$

So there is no component of the electric field pointing parallel to the wire; the field points radially away from the wire.

$$\vec{E} = \frac{2k\lambda}{r}\,\hat{r}$$

This matches the result found by taking the gradient of the potential.

d) Find the electric field around a finite, uniformly charged, straight wire, at a point a distance r straight out from the midpoint, starting from Coulomb's Law.

Solution:

I've already solved for this in the previous part (before I took the limit).

$$
\begin{aligned}
\vec{E} &= \frac{2Lk\lambda\left(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}\right)}{r\,\left(r^2 + L^2\right)^{1/2}} \\
&= \frac{2Lk\lambda\,\hat{r}}{r\,\left(r^2 + L^2\right)^{1/2}}
\end{aligned}
$$