Symmetries and Idealizations Homework 4

Due Friday 10/9

Problem 4.1 Surface of a cone (practice) Using integration, find the surface area of a cone.

Problem 4.2 Potential of a cylinder (practice) Consider a finite (hollow) cylinder of charge with uniform surface charge density σ on its sides (but not its ends).

a) Find the electric potential on the axis of symmetry, far from the cylinder. Please keep the first two non-zero terms.

b) Does your answer make sense? Why?

c) What ratio of height to width would make this cylinder look most like a point charge, when measured along its axis of symmetry?

Solution to problem 4.2 Potential of a cylinder

a) We showed in class that the potential on the axis due to a ring of charge is given by

$$V(z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{R^2+z^2}}$$

Chop up the cylinder into thin rings at height z', with thickness dz'. Each such ring has total charge $\sigma 2\pi R\,dz'$, which can be inserted into the result above, noting that the vertical distance to this ring is now $|z-z'|$. The total potential due to the cylinder is therefore

$$V(z) = \int_{-h/2}^{h/2} \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi R\,dz'}{\sqrt{R^2+(z-z')^2}}$$

where we have assumed that the cylinder has height h and is centered on the xy-plane. This integral could be done using a hyperbolic trig substitution of the form $z - z' = R\sinh\beta$ (or looked up in a book, or solved by Maple), but since we're only asked about the large-z limit, we don't need to actually solve it.

Instead, let's ask ourselves what is the small quantity. z is large, so it seems that R/z, z'/z and h/z are all small quantities, so

$$V(z) = \int_{-h/2}^{h/2} \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi R\,dz'}{\sqrt{R^2+(z-z')^2}} \tag{8.4}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\int_{-h/2}^{h/2} \frac{dz'}{\sqrt{R^2+(z-z')^2}} \tag{8.5}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\int_{-h/2}^{h/2}\frac{dz'}{\sqrt{\left(\frac{R}{z}\right)^2+\left(1-\frac{z'}{z}\right)^2}} \tag{8.6}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\int_{-h/2}^{h/2}\frac{dz'}{\sqrt{1+\left(\frac{R}{z}\right)^2-2\frac{z'}{z}+\frac{z'^2}{z^2}}} \tag{8.7}$$

$$u \equiv \frac{R^2}{z^2}-2\frac{z'}{z}+\frac{z'^2}{z^2} \tag{8.8}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\int_{-h/2}^{h/2}\left(1-u+u^2-u^3+\cdots\right)dz' \tag{8.9}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\int_{-h/2}^{h/2}\left(1-\left(\frac{R^2}{z^2}-2\frac{z'}{z}+\frac{z'^2}{z^2}\right)+\left(\frac{R^2}{z^2}-2\frac{z'}{z}+\frac{z'^2}{z^2}\right)^2-\left(\frac{R^2}{z^2}-2\frac{z'}{z}+\frac{z'^2}{z^2}\right)^3+\cdots\right)d \tag{8.10}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\int_{-h/2}^{h/2}\left(1-\frac{R^2}{z^2}-2\frac{z'}{z}+\frac{z'^2}{z^2}+\frac{R^4}{z^4}-4\frac{R^2z'}{z^3}+2\frac{R^2z'^2}{z^4}+4\frac{z'^2}{z^2}-4\frac{z'^3}{z^3}+\frac{z'^4}{z^4}+\cdots\right)dz' \tag{8.11}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\left(z'-\frac{R^2}{z^2}z'-\frac{z'^2}{z}+\frac{z'^3}{3z^2}+\frac{R^4}{z^4}z'-2\frac{R^2z'^2}{z^3}+\frac{2}{3}\frac{R^2z'^3}{z^4}+\frac{4}{3}\frac{z'^3}{z^2}-\frac{z'^4}{z^3}+\frac{1}{5}\frac{z'^5}{z^4}+\cdots\right)_{-h/2}^{h/2} \tag{8.12}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\left(h-\frac{R^2}{z^2}h+\frac{h^3}{12z^2}+\frac{R^4}{z^4}h+\frac{1}{6}\frac{R^2h^3}{z^4}+\frac{1}{6}\frac{h^3}{z^2}+\frac{1}{80}\frac{h^5}{z^4}+\cdots\right) \tag{8.13}$$

We can now see that the first two non-zero terms will be of order $1/z$ and $1/z^3$, so we can drop the higher-order terms. In fact, if we wanted higher order, we would have had to keep even higher-order terms in the u expansion.

$$V(z) = \frac{\sigma 2\pi R}{4\pi\epsilon_0}\frac{1}{|z|}\left(h-\frac{R^2}{z^2}h+\frac{h^3}{12z^2}+\frac{1}{6}\frac{h^3}{z^2}+\cdots\right) \tag{8.14}$$

$$= \frac{\sigma 2\pi R}{4\pi\epsilon_0}\left(\frac{h}{|z|}+\frac{h^3-R^2}{4|z|^3}+\cdots\right) \tag{8.15}$$

$$\tag{8.16}$$

b) Yes, my answer does make sense. The first term is just the potential due to a point charge with the same total charge (surface density times area).

Only odd powers survive because we've got a symmetric charge density.

The second term I can't figure out in my head exactly, but it does make sense that the bigger h is, the closer the cylinder will extend towards the point I'm considering, and the larger R is, the farther away it will be. So the signs make sense to me.

c) If we want the cylinder to behave like a point charge, we should make the second term disappear. We can do this by setting $h = R$ so its height is the same as its radius.

Problem 4.3 Cosmic Asimov You are part of the team building *Cosmic AC*, Asimov's ultimate, universe-sized computer. Your job is to fabricate a charged disk, 10 meters in radius and 1 cm thick. The charge density on the disk should be:

$$\rho = \alpha e^{-\beta r^2} \cos(\gamma z)$$

a) What is the total charge on the disk, in terms of the parameters α, β, and γ?

Solution:

$$\begin{aligned}
Q &= \int \rho \, dV \\
&= \int_0^{2\pi} \int_0^R \int_{z_1}^{z_2} \alpha \, e^{-\beta r^2} (\cos \gamma z) \, r \, dr \, d\phi \, dz \\
&= \alpha \left[\int_0^{2\pi} d\phi \right] \left[\int_0^R e^{-\beta r^2} \, r \, dr \right] \left[\int_{z_1}^{z_2} \cos \gamma z \, dz \right] \\
&= \alpha 2\pi \left[-\frac{e^{-\beta r^2}}{2\beta} \Bigg|_0^R \right] \left[\frac{1}{\gamma} \sin \gamma z \Bigg|_{z_1}^{z_2} \right] \\
&= 2\pi\alpha \, \frac{1}{2\beta} \left(1 - e^{-\beta R^2} \right) \frac{1}{\gamma} \left(\sin \gamma z_2 - \sin \gamma z_1 \right)
\end{aligned}$$

b) What are the dimensions of α, β, and γ?

Solution:

$\alpha \sim \frac{Q}{L^3}$ since the special functions are dimensionless and the whole expression must have dimensions of charge density.

$\beta \sim \frac{1}{L^2}$ since the argument of the exponential function must be dimensionless.

$\gamma \sim \frac{1}{L}$ since the argument of the cosine function must be dimensionless.

c) Design specifications indicate that: the maximum charge density should be $27\ \frac{\text{C}}{\text{cm}^3}$, only one-half period of the $\cos(\gamma z)$ term spans the whole height of the disk, the upper and lower circular surfaces are to have zero charge density, and the maximum values of the charge density on the circumference of the disk should be 10 percent of the maximum in the center. Find values for α, β, and γ.

Solution:

$\alpha = 27 \frac{C}{m^3}$

Determine β from the condition:

$$e^{-\beta R^2} = .1 \qquad \Rightarrow \qquad \beta = -\frac{1}{(10m)^2} \ln 0.1 = 0.023\, m^{-2}$$

Determine γ from the conditions: $\cos\gamma z_1 = 0$, $\cos\gamma z_2 = 0$, and $z_2 - z_1 = 1$ cm. Therefore:

$$\gamma = \frac{\pi}{2} \frac{1}{\frac{1}{2}\text{cm}} = \pi \text{cm}^{-1}$$

d) What is the total charge on the disk?

Solution:

You can plug in the numbers.

e) Estimate how much error your would make in your calculation of the total charge if you assumed that the disk was infinitely wide.

Solution:

If you assume the disk is infinitely wide, then the r integral runs from 0 to ∞ instead of from 0 to R. The percentage error is:

$$\begin{aligned} \text{Error} &= \frac{\left(\int_0^\infty e^{-\beta r^2} r\, dr\right) - \left(\int_0^R e^{-\beta r^2} r\, dr\right)}{\left(\int_0^\infty e^{-\beta r^2} r\, dr\right)} \\ &= 1 - \left(1 - e^{-\beta R^2}\right) \\ &= e^{-\beta R^2} \end{aligned}$$

$$= \quad 0.1$$

f) What is the surface charge density of the disk?

Solution:

To find the surface charge density on the disk, you need to integrate the z-dependence.

$$\begin{aligned}
\sigma &= \int_{z_1}^{z^2} \rho \, dz \\
&= \int_{z_1}^{z^2} \alpha e^{-\beta r^2} \cos(\gamma z) \, dz \\
&= 2\alpha e^{-\beta r^2}
\end{aligned}$$

Problem 4.4 Finite disk

a) Starting with the integral expression for the electrostatic potential due to a ring of charge, find the value of the potential everywhere along the axis of symmetry.

Solution:

Along the axis of symmetry, $r = 0$, and the integral becomes

$$\begin{aligned}
V(z) &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda \, R \, d\phi'}{\sqrt{R^2 + z^2}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{2\pi\lambda \, R}{\sqrt{R^2 + z^2}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{R^2 + z^2}}
\end{aligned}$$

b) Find the electrostatic potential everywhere along the axis of symmetry due to a finite disk of charge with uniform (surface) charge density σ. Start with your answer to part (a)

Solution:

The disk can be viewed as consisting of lots of rings of radius r' and width dr'. The charge on each such ring is

$$Q(r') = \sigma(2\pi r'\, dr')$$

and the potential for each such ring is, by part (a),

$$V(z) = \frac{1}{4\pi\epsilon_0}\frac{Q(r')}{\sqrt{r'^2 + z^2}}$$

Adding up all the rings which make up a disk of radius R, we obtain

$$\begin{aligned} V(z) &= \int_0^R \frac{\sigma}{4\pi\epsilon_0}\frac{2\pi r'\, dr'}{\sqrt{r'^2+z^2}} \\ &= \frac{\sigma}{4\pi\epsilon_0} 2\pi\sqrt{r'^2+z^2}\Big|_0^R \\ &= \frac{\sigma}{4\pi\epsilon_0}\left(2\pi\sqrt{R^2+z^2} - 2\pi z\right) \end{aligned}$$

c) Find two nonzero terms in a series expansion of your answer to part (b) for the value of the potential very far away from the disk.

Solution:

If $z \gg R$, then

$$\begin{aligned} \sqrt{R^2+z^2} &= z\sqrt{1+\frac{R^2}{z^2}} \\ &= z\left(1 + \frac{1}{2}\frac{R^2}{z^2} - \frac{1}{8}\frac{R^4}{z^4} + ...\right) \end{aligned}$$

Inserting this into the result obtained in (b) leads to

$$\begin{aligned} V(z) &= \frac{\sigma}{4\pi\epsilon_0}\left(2\pi\sqrt{R^2+z^2} - 2\pi z\right) \\ &= \frac{\sigma}{4\pi\epsilon_0}\left(\frac{\pi R^2}{z} - \frac{\pi R^4}{4z^3} + ...\right) \end{aligned}$$

Problem 4.5 Potential of a cone [1] A conical surface (an empty ice-cream cone) carries a uniform charge density σ. The height of the cone is a, as is the radius of the top. Find the potential at the point in the center of the opening of the cone), letting the potential at infinity be zero.

[1] based on GEM 2.27

Solution to problem 4.5 Potential of a cone The cone that I'll be considering has its vertex on the origin and opens up on the positive z-axis. I'll use cylindrical coordinates in my solution. Starting with the general expression for calculating potential:

$$V(\vec{\boldsymbol{r}}) = \int\int \frac{k\,\sigma\,dA'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}$$

then I'll use what I know. For a cone with height h and base radius s, the differential area element is:

$$dA' = \frac{s}{h^2}\sqrt{s^2+h^2}\,z\,dz\,d\phi$$

in this problem, the height and the radius of the base are the same value a, so the differential area element is:

$$dA' = \sqrt{2}\,z'\,dz'\,d\phi'$$

Using this differential area element, and rewriting the denominator is cylindrical coordinates, the potential becomes:

$$V(\vec{\boldsymbol{r}}) = \int_0^{2\pi}\int_0^a \frac{\sqrt{2}\,k\,\sigma\,z'\,dz'\,d\phi'}{\sqrt{r^2 + r'^2 - 2rr'\cos(\phi-\phi') + (z-z')^2}}$$

Now, simplifying the denominator, on the z-axis where the potential is evaluated, $z = a$, $r = 0$ and for our cone $r' = z'$:

$$\begin{aligned} V(z=a) &= \int_0^{2\pi}\int_0^a \frac{\sqrt{2}\,k\,\sigma\,z'\,dz'\,d\phi'}{\sqrt{z'^2 - +(a-z')^2}} \\ &= \int_0^{2\pi}\int_0^a \frac{\sqrt{2}\,k\,\sigma\,z'\,dz'\,d\phi'}{\sqrt{2z'^2 - 2zz' + a^2}} \end{aligned}$$

Using Maple or an integral table to evaluate the integral:

$$V(z=a) = k\,\sigma\,\frac{a\sqrt{2}}{4}\,\ln\left[\frac{2+\sqrt{2}}{2-\sqrt{2}}\right]$$