Symmetries and Idealizations Homework 3

Due Wednesday 10/7

Problem 3.1 Ice cream mass (practice) Use integration to find the total mass of ice cream in a packed cone (both cone and hemisphere of ice cream on top).

Problem 3.2 Total charge (practice) For each case below, find the total charge.

a) A positively charged (dielectric) spherical shell of inner radius a and outer radius b with a spherically symmetric internal charge density $\rho(\vec{\boldsymbol{r}}) = \alpha\, 3e^{(kr)^3}$

b) A positively charged (dielectric) cylindrical shell of inner radius a and outer radius b with a cylindrically symmetric internal charge density $\rho(\vec{\boldsymbol{r}}) = \alpha\, \frac{1}{r}\, e^{kr}$.

Solution to problem 3.2 Total charge In both cases, the charge density depends only on r, and is zero for $r < a$ or $r > b$. We therefore want to multiply the given charge density by a suitable combination of step functions, which turns out to be $\Theta(r-a) - \Theta(r-b)$. (There are other ways of writing this combination.)

Case (a) The charge density everywhere is given by

$$\rho(r) = \alpha\, 3e^{(kr)^3}\Big(\Theta(r-a) - \Theta(r-b)\Big)$$

The total charge is given by

$$\int\limits_{\text{shell}} \rho(\vec{\boldsymbol{r}})d\tau = \int_0^{2\pi}\int_0^{\pi}\int_a^{b} \alpha\, 3e^{(kr)^3} r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{4\pi\alpha}{k^3}\, e^{k^3r^3}\Bigg|_a^b$$

Note that k has the dimensions of inverse length, and α has the dimensions of charge per length cubed.

Case (b) The charge density everywhere is given by

$$\rho(r) = \alpha\, \frac{1}{r}\, e^{kr}\Big(\Theta(r-a) - \Theta(r-b)\Big)$$

This problem is ill-posed, since the height of the cylinder is not given. Assuming that the cylinder extends from $z = A$ to $z = B$, the total charge is given by

$$\int\limits_{\text{shell}} \rho(\vec{\boldsymbol{r}})d\tau = \int_A^{B}\int_0^{2\pi}\int_a^{b} \alpha\, \frac{1}{r}\, e^{kr} r\, dr\, d\phi\, dz = \frac{2\pi\alpha}{k}(B-A)\, e^{kr}\Bigg|_a^b$$

Note that α now has the dimensions of charge per length squared.

If the cylinder is infinitely long, then the total charge is of course also infinite. But a better answer in this case would be that the charge per unit length is $\frac{2\pi\alpha}{k}\left(e^{kb} - e^{ka}\right)$.

Problem 3.3 Quadrupole

a) A linear quadrupole is a series of three charges in a line, in this case, along the z-axis. Charges $+Q$ at $z = \pm D$ and charge $-2Q$ at $z = 0$. Find the electrostatic potential at a point P in the x,y-plane at a distance r from the center of the quadrupole.

Solution:

Choose the point at which you are trying to measure the potential to lie on the x-axis or choose to work in cylindrical coordinates. By the principle of superposition, the potential due to the three charges is just the sum of the potentials due to the individual charges.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{|\vec{r}-\vec{r'}|} - \frac{2Q}{|\vec{r}-\vec{r''}|} + \frac{Q}{|\vec{r}-\vec{r'''}|}\right\} \tag{8.4}$$

$$= \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{\left|r\hat{\imath}-D\hat{k}\right|} - \frac{2Q}{r} + \frac{Q}{\left|r\hat{\imath}+D\hat{k}\right|}\right\} \tag{8.5}$$

$$= \frac{2Q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+D^2}} - \frac{1}{r}\right\} \tag{8.6}$$

b) Assume $r \gg D$. Find the first two non-zero terms of a Laurent series expansion to the electrostatic potential you found in the first part of this problem.

Solution:

In the first term from your answer to part (a) above, pull out a factor of r^2 from the square root. Then use the memorized series expansion for $(1+\epsilon)^p$ with $\epsilon = \frac{D}{r}$ and $p = \frac{1}{2}$.

$$V(\vec{r}) = \frac{2Q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+D^2}} - \frac{1}{r}\right\} \tag{8.7}$$

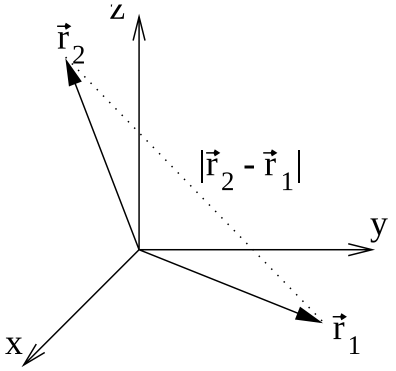
$$= \frac{2Q}{4\pi\epsilon_0}\left\{\frac{1}{r\sqrt{1+\left(\frac{D}{r}\right)^2}} - \frac{1}{r}\right\} \tag{8.8}$$

$$= \frac{2Q}{4\pi\epsilon_0}\left\{\frac{1}{r}\left(1 - \frac{1}{2}\left(\frac{D}{r}\right)^2 + \frac{3}{8}\left(\frac{D}{r}\right)^4 + \dots\right) - \frac{1}{r}\right\} \tag{8.9}$$

$$= \frac{2Q}{4\pi\epsilon_0}\left\{\frac{1}{r}\left(-\frac{1}{2}\left(\frac{D}{r}\right)^2 + \frac{3}{8}\left(\frac{D}{r}\right)^4 + \dots\right)\right\} \tag{8.10}$$

Problem 3.4 Curvilinear distance

a) Find the distance $|\vec{r}_2 - \vec{r}_1|$ between the point $\vec{r}_1 = (x_1, y_1, z_1)$ and the point $\vec{r}_2 = (x_2, y_2, z_2)$ in rectangular coordinates.



Solution:

In rectangular coordinates:

$$|\vec{r}_2 - \vec{r}_1| = \left|(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})\right| \tag{8.11}$$

$$\tag{8.12}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{8.13}$$

b) Show that this same distance written in cylindrical coordinates is:

$$|\vec{r}_2 - \vec{r}_1| = \sqrt{r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$$

Solution:

In cylindrical coordinates:

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned}$$

Plug these coordinates into the answer to part (a) above. (Note: It is nontrivial to start this calculation from the position vector. Why? What is the position vector $\vec{r}$ in cylindrical coordinates?)

$$|\vec{r}_2 - \vec{r}_1| = \sqrt{(r_2\cos\phi_2 - r_1\cos\phi_1)^2 + (r_2\sin\phi_2 - r_1\sin\phi_1)^2 + (z_2 - z_1)^2}$$

$$\begin{aligned} &= \left[r_2^2(\cos^2\phi_2+\sin^2\phi_2)+r_1^2(\cos^2\phi_1+\sin^2\phi_1)\right.\\ &\qquad \left.-2r_1r_2(\cos\phi_2\cos\phi_1+\sin\phi_2\sin\phi_1)+(z_2-z_1)^2\right]^{\frac{1}{2}}\\ &= \sqrt{r_2^2+r_1^2-2r_1r_2\cos(\phi_2-\phi_1)+(z_2-z_1)^2} \end{aligned}$$

c) Show that this same distance written in spherical coordinates is:

$$|\vec{r}_2-\vec{r}_1| = \sqrt{r_2^2+r_1^2-2r_1r_2\left[\sin\theta_2\sin\theta_1\cos(\phi_2-\phi_1)+\cos\theta_2\cos\theta_1\right]}$$

Solution:

In spherical coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\phi\\ y &= r\sin\theta\sin\phi\\ z &= r\cos\theta \end{aligned}$$

Plug these coordinates into the answer to part (a) above. (Note: It is nontrivial to start this calculation from the position vector. Why? What is the position vector $\vec{r}$ in spherical coordinates?)

$$\begin{aligned} |\vec{r}_2-\vec{r}_1| &= \left[(r_2\sin\theta_2\cos\phi_2-r_1\sin\theta_1\cos\phi_1)^2\right.\\ &\qquad +(r_2\sin\theta_2\sin\phi_2-r_1\sin\theta_1\sin\phi_1)^2\\ &\qquad \left.+(r_2\cos\theta_2-r_1\cos\theta_1)^2\right]^{\frac{1}{2}}\\ &= \left\{r_2^2[\sin^2\theta_2(\cos^2\phi_2+\sin^2\phi_2)+\cos^2\theta_2]\right.\\ &\qquad +r_1^2[\sin^2\theta_1(\cos^2\phi_1+\sin^2\phi_1)+\cos^2\theta_1]\\ &\qquad \left.-2r_1r_2[\sin\theta_2\sin\theta_1(\cos\phi_2\cos\phi_1+\sin\phi_2\sin\phi_1)+\cos\theta_2\cos\theta_1]\right\}^{\frac{1}{2}}\\ &= \sqrt{r_2^2+r_1^2-2r_1r_2[\sin\theta_2\sin\theta_1\cos(\phi_2-\phi_1)+\cos\theta_2\cos\theta_1]} \end{aligned}$$

d) Now assume that $\vec{r}_1$ is in the x-y plane. Simplify the previous two formulas.

Solution:

Cylindrical Coordinates at $z_1=0$

$$\sqrt{r_2^2+r_1^2-2r_1r_2\cos(\phi_2-\phi_1)+z_2^2} \tag{8.14}$$

Spherical Coordinates at $\theta_1 = \frac{\pi}{2} \Rightarrow \cos\theta_1 = 0$ and $\sin\theta_1 = 1$.

$$\sqrt{r_2^2 + r_1^2 - 2r_1 r_2 \sin\theta_2 \cos(\phi_2 - \phi_1)} \tag{8.15}$$

e) **CHALLENGE:**[1] Find the distance $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ between the point $\vec{\boldsymbol{r}}$ and the point $\vec{\boldsymbol{r}}'$ in terms of the magnitudes of $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}'$ and γ, the angle between them. (Do **not** choose a coordinate system.) Then assuming that $\vec{\boldsymbol{r}} \gg \vec{\boldsymbol{r}}'$, find a series expansion for $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$, correct to fourth order. This expansion is the basis of multipole expansions, used in both electromagnetic theory and quantum mechanics.

Problem 3.5 Mass of a slab Determine the total mass of each of the slabs below.

a) A square slab of side length L with thickness h, resting on a table top at $z = 0$, whose mass density is given by $\rho = A\pi\sin(\pi z/h)$.

Solution:

$$\begin{aligned} M &= \int\limits_{\text{slab}} \rho\, dV \\ &= L^2 \int\limits_0^h A\pi \sin(\pi z/h)\, dz \\ &= 2AhL^2 \end{aligned}$$

b) A square slab of side length L with thickness h, resting on a table top at $z = 0$, whose mass density is given by

$$\rho = 2A\Big(\Theta(z) - \Theta(z-h)\Big) \tag{8.16}$$

Solution:

$$\begin{aligned} M &= \int\limits_{\text{slab}} \rho\, dV \\ &= L^2 \int\limits_{-\infty}^{\infty} 2A\Big(\Theta(z) - \Theta(z-h)\Big)\, dz \end{aligned}$$

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[1] not required, but give it a try if you are not overwhelmed with the course.

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\begin{aligned}
&= L^2 \int\limits_0^h 2A\,dz \\
&= 2AhL^2
\end{aligned}
$$

c) An infinitesimally thin square sheet of side length L, resting on a table top at $z = 0$, whose surface density is given by $\sigma = 2Ah$.

Solution:

$$
\begin{aligned}
M &= \int\limits_{\text{slab}} \rho\,dV \\
&= L^2 \int\limits_{-\infty}^{\infty} 2Ah\,\delta(z)\,dz \\
&= 2AhL^2
\end{aligned}
$$

d) An infinitesimally thin square sheet of side length L, resting on a table top at $z = 0$, whose mass density is given by $\rho = 2Ah\,\delta(z)$.

Solution:

Since the surface density is constant, we simply multiply by the area of the slab, obtaining

$$M = \sigma L^2 = 2AhL^2$$

e) Write several sentences comparing your answers. What units does A have?

Solution:

All of these slabs have the same mass. The second can be viewed as an approximation to the first, in which the internal structure of the slab doesn't matter. The last two are further idealizations of the second.