Symmetries and Idealizations Homework 2

Due Friday 10/2

Problem 2.1 Series notation I (practice) Write out the first four nonzero terms in the series:

a)

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

Solution:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + ... \end{aligned}$$

b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

Solution:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} &= \frac{-1^1}{1!} + \frac{-1^2}{2!} + \frac{-1^3}{3!} + \frac{-1^4}{4!} + ... \\ &= -1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} + ... \end{aligned}$$

Problem 2.2 Series notation II (practice) Write the following series using sigma ($\sum$) notation.

a)

$$1 - 2\,\theta^2 + 4\,\theta^4 - 8\,\theta^6 + \ \ldots$$

Solution:

$$\sum_{n=0}^{\infty} -2^n\,\theta^{2n}$$

b)

$$\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \ \ldots$$

Solution:

$$\sum_{n=2}^{\infty} \frac{-1^n}{n^2}$$

Problem 2.3 Series notation III (practice) If you need more practice with sigma ($\sum$) notation, you can get really good practice by going back and forth between the two representations of the standard power series on the memorization page. Power series are used everywhere in physics and it is very important to be able to translate back and forth between the two representations.

Problem 2.4 Plotting a vector relation

a) Make a sketch of the graph

$$|\vec{r} - \vec{a}| = 2 \tag{4.50}$$

for each of the following values of $\vec{a}$:

$$\vec{a} = \vec{0} \tag{4.51}$$

$$\vec{a} = 2\hat{\imath} - 3\hat{\jmath} \tag{4.52}$$

$$\vec{a} = \text{ points due east and is 2 units long} \tag{4.53}$$

Solution:

These are the equations for a circle of radius 2 centered at (i) the origin, (ii) $(2,3)$ and (iii)$(2,0)$, respectively.

b) Derive a more familiar equation equivalent to

$$|\vec{r} - \vec{a}| = 2 \tag{4.54}$$

for arbitrary $\vec{a}$, by expanding $\vec{r}$ and $\vec{a}$ in rectangular coordinates. Simplify as much as possible. (Ok, ok, I know this is a terribly worded question. What do I mean by "more familiar"? What do I mean by "simplify as much as possible"? Why am I making you read my mind? Try it anyway. Real life is not full of carefully worded problems. Bonus points to anyone who can figure out a better way of wording the question that doesn't give the point away.)

Solution:

Let

$$\vec{r} = r_x\hat{i} + r_y\hat{j}$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

then

$$\begin{aligned}
|\vec{r}-\vec{a}| &= \sqrt{(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{a})} \\
&= \sqrt{[(r_x\hat{i}+r_y\hat{j})-(a_x\hat{i}+a_y\hat{j})]\cdot[(r_x\hat{i}+r_y\hat{j})-(a_x\hat{i}+a_y\hat{j})]} \\
&= \sqrt{(r_x-a_x)^2+(r_y-a_y)^2}
\end{aligned}$$

So:

$$\begin{aligned}
|\vec{r}-\vec{a}| &= 2 \\
\rightarrow \sqrt{(r_x-a_x)^2+(r_y-a_y)^2} &= 2 \\
(r_x-a_x)^2+(r_y-a_y)^2 &= 4
\end{aligned}$$

which is the equation for a circle written in Cartesian coordinates.

c) Write a brief description of the geometric meaning of the equation

$$|\vec{r}-\vec{a}| = 2 \tag{4.55}$$

Solution:

This is the equation for a circle centered at the tip of vector $\vec{a}$ with radius 2 units.

Problem 2.5 Tetrahedron Find the angle between any two line segments that join the center of a tetrahedron to its vertices. Hint: Think of the vertices of the tetrahedron as sitting at the vertices of a cube. (You may need to build a model and play with it to see how this works!)

Solution:

Think of one of the vertices of the tetrahedron lying at the origin, and the other vertices at $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$. The center of the tetrahedron is at $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Consider two vectors each pointing from the center of the tetrahedron to one of the vertices (say the first two on the list above). Then we can calculate the dot product of these two vectors in two ways: algebraically and geometrically.

$$\left((\hat{\imath}+\hat{\jmath})-(\frac{1}{2}\hat{\imath}+\frac{1}{2}\hat{\jmath}+\frac{1}{2}\hat{k})\right) \cdot \left((\hat{\imath}+\hat{k})-(\frac{1}{2}\hat{\imath}+\frac{1}{2}\hat{\jmath}+\frac{1}{2}\hat{k})\right)$$

$$
\begin{aligned}
&= \left|\frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} - \frac{1}{2}\hat{k}\right| \left|\frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k}\right| \cos\gamma \\
\left(\frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} - \frac{1}{2}\hat{k}\right) \cdot \left(\frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k}\right) &= \sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}} \cos\gamma \\
-\frac{1}{4} &= \frac{3}{4}\cos\gamma \\
\cos\gamma &= -\frac{1}{3} \\
\gamma &= \cos^{-1} -\frac{1}{3} \\
&= \approx 109^o
\end{aligned}
$$

Problem 2.6 Series convergence Recall that, if you take an infinite number of terms, the series for $\sin z$ and the function itself $f(z) = \sin z$ are equivalent representations of the same thing for all real numbers z, (in fact, for all complex numbers z). This is not always true. More commonly, a series is only a valid, equivalent representation of a function for some more restricted values of z. The technical name for this idea is convergence–the series only "converges" to the value of the function on some restricted domain.

Find the power series for the function $f(z) = \frac{1}{1+z^2}$. Then, using the Maple worksheet from class `http://www.physics.oregonstate.edu/ph320/maple/powerapprox.mws` [1] as a model, explore the convergence of this series. Where does your series for this new function converge? Can you see the reflection of the region of convergence in the graphs of the various approximations? Print out a plot and write a brief description (a sentence or two) of the region of convergence.

Solution to problem 2.6 Series convergence See the Maple worksheet `http://physics.oregonstate` which can be found as a link on the class syllabus.

Problem 2.7 Euler's formula (challenge)

a) Work out the power series for e^{ix}.

b) What is the real part of this power series?

c) What is the imaginary part of this power series?

d) How is e^{ix} related to $\sin x$ and $\cos x$?

Solution to problem 2.7 Euler's formula

a) The power series for e^{ix} is

$$e^{ix} = 1 + ix - \frac{1}{2!}x^2 - \frac{i}{3!}x^3 + \frac{1}{4!}x^4 - \frac{i}{5!}x^5 - \cdots$$

[1] the link is called "Approximating functions with a power series"

b) The real part is just

$$e^{ix} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

c) The imaginary part is just

$$e^{ix} = x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \cdots$$

d) A quick glance at the power series for $\cos x$ and $\sin x$ shows that

$$e^{ix} = \cos x + i \sin x$$