

## Calculating line elements in spherical and cylindrical coordinates

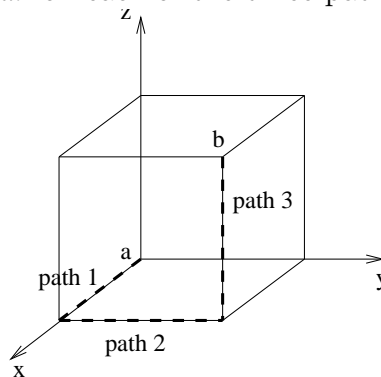
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### Rectangular Coordinates:

The arbitrary infinitesimal displacement vector in Cartesian coordinates is:

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

Given the cube shown below, find  $d\vec{r}$  on each of the three paths, leading from  $a$  to  $b$ .

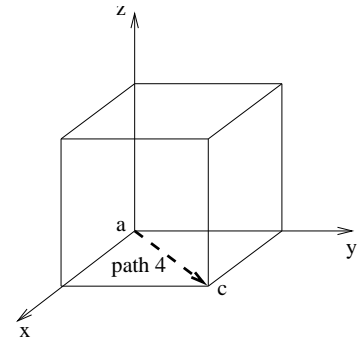


Path 1:  $d\vec{r} =$

Path 2:  $d\vec{r} =$

Path 3:  $d\vec{r} =$

The first expression above for  $d\vec{r}$  is valid for any path in rectangular coordinates. Find the appropriate expression for  $d\vec{r}$  for the path which goes directly from  $a$  to  $c$  as drawn below.



Path 4:  $d\vec{r} =$

However, Cartesian coordinates would be a **poor** choice to describe a path on a cylindrically or spherically shaped surface. Next we will find an appropriate expression in these cases.

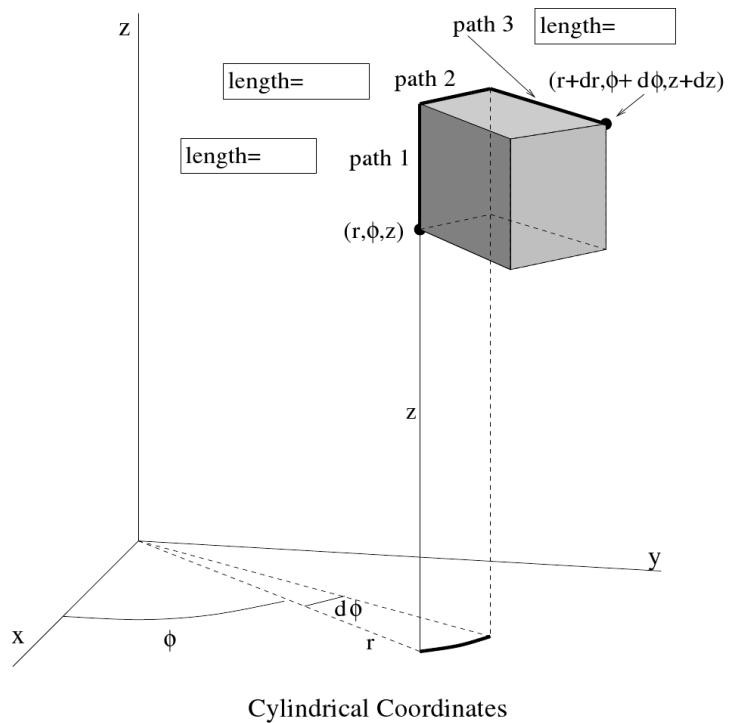
## Cylindrical Coordinates:

You will now derive the general form for  $d\vec{r}$  in cylindrical coordinates by determining  $d\vec{r}$  along the specific paths below.

Note that an infinitesimal element of length in the  $\hat{r}$  direction is simply  $dr$ , just as an infinitesimal element of length in the  $\hat{i}$  direction is  $dx$ . **But**, an infinitesimal element of length in the  $\hat{\phi}$  direction is **not** just  $d\phi$ , since this would be an angle and does not even have the units of length.

Geometrically determine the length of the three paths leading from  $a$  to  $b$  and write these lengths in the corresponding boxes on the diagram.

Now, remembering that  $d\vec{r}$  has both magnitude and direction, write down below the infinitesimal displacement vector  $d\vec{r}$  along the three paths from  $a$  to  $b$ . Notice that, along any of these three paths, only one coordinate  $r$ ,  $\phi$ , or  $z$  is changing at a time. (i.e. along path 1,  $dz \neq 0$ , but  $d\phi = 0$  and  $dr = 0$ ).



Path 1:  $d\vec{r} =$

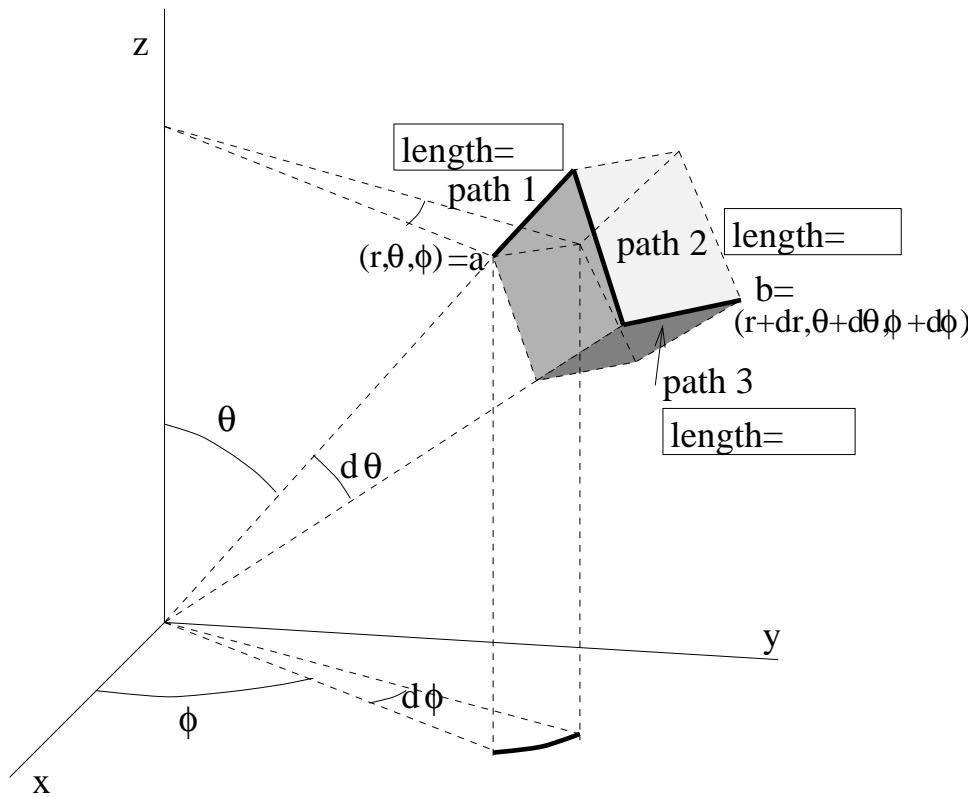
Path 2:  $d\vec{r} =$

Path 3:  $d\vec{r} =$

If all three coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this  $d\vec{r}$  for any path as:

$$d\vec{r} =$$

This is the general line element in cylindrical coordinates.



Line Elements in Spherical Coordinates

### Spherical Coordinates:

You will now derive the general form for  $d\vec{r}$  in spherical coordinates by determining  $d\vec{r}$  along the specific paths below. As in the cylindrical case, note that an infinitesimal element of length in the  $\hat{\theta}$  or  $\hat{\phi}$  direction is **not** just  $d\theta$  or  $d\phi$ . You will need to be more careful. Geometrically determine the length of the three paths leading from  $a$  to  $b$  and write these lengths in the corresponding boxes on the diagram. Now, remembering that  $d\vec{r}$  has both magnitude and direction, write down below the infinitesimal displacement vector  $d\vec{r}$  along the three paths from  $a$  to  $b$ . Notice that, along any of these three paths, only one coordinate  $r$ ,  $\theta$ , or  $\phi$  is changing at a time. (i.e. along path 1,  $d\theta \neq 0$ , but  $dr = 0$  and  $d\phi = 0$ ).

Path 1:  $d\vec{r} =$

Path 2:  $d\vec{r} =$

(Be careful, this is the tricky one.)

Path 3:  $d\vec{r} =$

If all 3 coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this  $d\vec{r}$  for any path as:

$d\vec{r} =$

This is the general line element in spherical coordinates.