Central Forces Homework 7

Due 5/29/19, 4 pm

PRACTICE:

1. **Quantum Particle in a 2-D Box**
   You know that the spatial eigenfunctions for a particle in a 1-D box of length $L = \pi$ are $\sin nx$. If you want the eigenfunctions for a particle in a 2-D box, then you just multiply together the eigenfunctions for a 1-D box in each direction.

   (a) Find the eigenfunctions for a particle in a 2-D box with sides of length L_1 in the x-direction and length L_2 in the y-direction.

   (b) Any sufficiently smooth spatial wave function inside a 2-D box can be expanded in a **double** sum of the product wave functions, i.e.

   $$\psi(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\ \text{eigenfunction}_n(x)\ \text{eigenfunction}_m(y)$$

   Using your expressions from part (a) above, write out all the terms in this sum out to $n = 3$, $m = 3$.

   (c) Find a formula for the c_{nm}s in part (b).

REQUIRED:

2. (2 points each)

   Answer the following questions for a quantum mechanical particle confined to a ring. You may want to use the Mathematica code (cfqmring.nb) on time dependence on the ring to help you figure out the answers.

   (a) Characterize the states for which the probability density does not depend on time.

   (b) Characterize the states that are right-moving.

   (c) Characterize the states that are standing waves.

   (d) Compare the time dependence of the three states:

   $$\begin{aligned}
   |\Psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle + |-3\rangle\right)\\
   |\Psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle - |-3\rangle\right)\\
   |\Psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle + i|-3\rangle\right)
   \end{aligned}$$

3. (2 points each)

   In this problem, you will carry out calculations on the following normalized abstract quantum state on a ring:

$$|\Psi\rangle = \sqrt{\frac{1}{4}}\left(|1\rangle + \sqrt{2}\,|2\rangle + |3\rangle\right)$$

   (a) You carry out a measurement to determine the energy of the particle at time $t = 0$. Calculate the probability that you measure the energy to be $\frac{4\hbar^2}{2I}$.

   (b) You carry out a measurement to determine the z-component of the angular momentum of the particle at time $t = 0$. Calculate the probability that you measure the z-component of the angular momentum to be $3\hbar$.

   (c) You carry out a measurement on the location of the particle at time, $t = 0$. Calculate the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{2}$.

   (d) You carry out a measurement to determine the energy of the particle at time $t = \frac{2I}{\hbar}\frac{\pi}{4}$. Calculate the probability that you measure the energy to be $\frac{4\hbar^2}{2I}$.

   (e) You carry out a measurement to determine the z-component of the angular momentum of the particle at time $t = \frac{2I}{\hbar}\frac{\pi}{4}$. Calculate the probability that you measure the z-component of the angular momentum to be $3\hbar$.

   (f) You carry out a measurement on the location of the particle at time, $t = \frac{2I}{\hbar}\frac{\pi}{4}$. Calculate the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{2}$.

   (g) Write a short paragraph explaining what representation/basis you used for each of the calculations above and why you chose to use that representation/basis.

   (h) In the above calculations, you should have found some of the quantities to be time dependent and others to be time independent. Briefly explain why this is so. That is, for a time dependent state like $|\Psi\rangle$ explain what makes some observables time dependent and others time independent.

4. (2 points each)

   Legendre Polynomials

   (a) Use *Mathematica* to find the first 5 Legendre polynomials.

   (b) Use Rodrigues' formula to calculate the first 5 Legendre polynomials. (You are encouraged to use *Mathematica* to help with the derivatives.

   (c) Look up two recurrence relations for Legendre polynomials and use them to find $P_3(z)$ and $P_3'(z)$ assuming that all you know is that $P_0(z) = 1$ and $P_1(z) = z$. Do this part of the problem by hand.

5. (8 points)

   Use your favorite tool (*e.g.* Maple, Mathematica, Matlab, pencil) to generate the Legendre polynomial expansion to the function $f(z) = \sin(\pi z)$. How many terms do you need to include in a partial sum to get a "good" approximation to $f(z)$ for $-1 < z < 1$? What do you mean by a "good" approximation? How about the interval $-2 < z < 2$? How good is your approximation? Discuss your answers. Answer the same set of questions for the function $g(z) = \sin(3\pi z)$

6. Heads-up: This homework problem will be due in the next homework set. But you should start it NOW. It requires many steps, which will be demonstrated for you in class (for a different example) over the next few days. If you do each step as we go along, it will be straightforward to complete. If you wait until the last minute you will NOT learn much. It may help you to complete the practice problem above. Bring questions about the practice problem to class on Tuesday.

   QM Particle in a 2-D Box
   Find an exact expression for the wave function for a quantum mechanical particle inside a 2-dimensional infinite potential well with sides of length L_x, L_y. The initial state is:

   $$\psi(x, y, 0) = \frac{30}{\sqrt{L_x^5 L_y^5}}(L_x x - x^2)(L_y y - y^2)$$

   I have chosen coordinates so that one of the corners of the box is at the origin and all of the box is in the first quadrant (i.e. all positive values of the spatial coordinates).

   Plot an approximation for the probability density at $t = 0$ and at an interesting later time. Explain why you chose the later time that you did. Explain how you chose your approximation scheme and why.