PH 422: Day 8

30 The Magnetic Field of a Straight Wire

Consider the magnetic field of a straight wire along the z-axis carrying a steady current $\vec{\boldsymbol{I}} = I\,\hat{\boldsymbol{z}}$. The Biot-Savart Law says that

$$\vec{\boldsymbol{B}} = -\frac{\mu_0 I}{4\pi\epsilon_0}\int_{-L}^{L}\frac{(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')\times d\vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^3}$$

and we have

$$\begin{aligned} \vec{\boldsymbol{r}} &= r\,\hat{\boldsymbol{r}} + z\,\hat{\boldsymbol{z}} \\ \vec{\boldsymbol{r}}' &= z'\,\hat{\boldsymbol{z}} \end{aligned}$$

so that

$$\begin{aligned} -(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')\times d\vec{\boldsymbol{r}} &= -\big(r\,\hat{\boldsymbol{r}} + (z-z')\,\hat{\boldsymbol{z}}\big)\times dz'\,\hat{\boldsymbol{z}} \\ &= r\,dz'\,\hat{\boldsymbol{\phi}} \end{aligned}$$

which gives the expected right-hand rule behavior for the direction of the magnetic field. We therefore have

$$\begin{aligned} \vec{\boldsymbol{B}} &= \frac{\mu_0 I}{4\pi\epsilon_0}\,r\,\hat{\boldsymbol{\phi}}\int_{-L}^{L}\frac{dz'}{\big(r^2+(z-z'^2)\big)^{3/2}} \\ &= -\frac{\mu_0 I}{4\pi\epsilon_0}\frac{(z-z')\,\hat{\boldsymbol{\phi}}}{\sqrt{r^2+(z-z'^2)}}\Bigg|_{-L}^{L} \end{aligned}$$

An important special case is an infinite wire — after all, a finite straight wire can not support a steady current. Taking the limit as $L\to\infty$, we obtain

$$\vec{\boldsymbol{B}} = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}$$

We will see next week that this result can be derived more simply using symmetry and Ampère's Law, an approach which fails however for the finite wire.