# PH 422: Day 7

## 28 Magnetic Fields

## 29 The Biot-Savart Law

We started with the superposition principle for the (electric) scalar potential

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}')\, d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

and the corresponding superposition principle for the electric field

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}')(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')\, d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

But we also have the relation

$$\vec{\boldsymbol{E}} = -\vec{\nabla} V$$

and we could have calculated the above integral representation of $\vec{\boldsymbol{E}}$ by taking the gradient of the one for $V$.

It is fairly easy to see that consistency of the above expressions requires

$$\vec{\nabla}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) = -\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

which can of course also be computed directly (and was part of the last homework assignment in the previous paradigm). Note that the gradient only "sees" $\vec{\boldsymbol{r}}$, not $\vec{\boldsymbol{r}}'$; we are only taking $x$, $y$, and $z$ derivatives (or their equivalents in curvilinear coordinates), not $x'$, $y'$, or $z'$ derivatives. This also means there is no difficulty bringing the gradient inside the integral sign. Direct verification of this identity is most easily done by writing everything out explicitly in rectangular coordinates.

Turning to the magnetic field, we have

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int \frac{\rho(\vec{\boldsymbol{r}}')\, \vec{\boldsymbol{v}}(r')\, d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} = \frac{\mu_0}{4\pi} \int \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')\, d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

and

$$\vec{\boldsymbol{B}} = \vec{\nabla} \times \vec{\boldsymbol{A}}$$

Can we compute an integral representation for $\vec{\boldsymbol{B}}$ by taking the curl of the one for $\vec{\boldsymbol{A}}$? Yes, moving curl inside the integral, and using the product rule for curl. The only dependence on $\vec{\boldsymbol{r}}$ is in the term on the denominator; that's "$f$". Everything else is "$\vec{\boldsymbol{G}}$", which is a constant vector field, so that the "$\vec{\nabla} \times \vec{\boldsymbol{G}}$" term is zero. Thus, all we need to do is use the previously derived formula for $\vec{\nabla}(1/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|)$, resulting in

$$\vec{\boldsymbol{B}} = \vec{\nabla} \times \vec{\boldsymbol{A}} = \frac{\mu_0}{4\pi} \int \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}') \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')\, d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

(where the minus sign has been removed by reversing the order of the cross product). This is known as the *Biot-Savart Law.*

For surface currents, the Biot-Savart Law takes the form

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi} \int \frac{\vec{\boldsymbol{K}}(\vec{\boldsymbol{r}}') \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')\, dA'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

and for line currents we have

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi} \int \frac{\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}') \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')\, ds'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} = -\frac{\mu_0 I}{4\pi} \int \frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}') \times d\vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

since $\vec{\boldsymbol{I}}\, ds' = I\, d\vec{\boldsymbol{r}}$. (The order of the cross product is often reversed in the last expression, but this requires the convention that the other factor still be regarded as being inside the integral.)