PH 422: Day 5

Please read Sections 2.3–2.5 and 3.11 from the mathematics notes.

26 Magnetic Vector Potential

Recall the use of the superposition principle to find the potential due to any charge distribution, starting from the electric potential due to a point charge, namely

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

resulting in

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\boldsymbol{r}}')\,d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

This is an example of a *Green function*, which encodes how space responds to a δ-function charge.

By analogy, there is a similar expression for the *(magnetic) vector potential*, starting from

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{q\,\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} = \frac{\mu_0}{4\pi}\frac{\vec{\boldsymbol{I}}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

and resulting in

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\int\frac{\rho(\vec{\boldsymbol{r}}')\,\vec{\boldsymbol{v}}\,d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} = \frac{\mu_0}{4\pi}\int\frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')\,d\tau'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

The constant μ_0 is called the *permeability of free space*, and plays a similar role for magnetic fields as does ϵ_0, the *permittivity of free space*, for electric fields.

Similar expressions hold for other current distributions. For a surface current, we have

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\int\frac{\sigma(\vec{\boldsymbol{r}}')\,\vec{\boldsymbol{v}}\,dA'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} = \frac{\mu_0}{4\pi}\int\frac{\vec{\boldsymbol{K}}(\vec{\boldsymbol{r}}')\,dA'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

and for a linear distribution we have

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\int\frac{\lambda(\vec{\boldsymbol{r}}')\,\vec{\boldsymbol{v}}\,ds'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} = \frac{\mu_0}{4\pi}\int\frac{\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}')\,ds'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

27 Ring of Current

Recall that for a thin current-carrying loop we have

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}')\,ds'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} = \frac{\mu_0}{4\pi}\int \frac{\lambda(\vec{\boldsymbol{r}}')\,\vec{\boldsymbol{v}}\,ds'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}$$

For a circular ring of current, we have

$$\vec{\boldsymbol{v}} = \frac{2\pi R}{T}\,\hat{\boldsymbol{\phi}} = \frac{2\pi R}{T}\left(-\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}}\right)$$

and

$$|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'| = \sqrt{r^2 - 2\,rR\cos(\phi-\phi') + R^2 + z^2}$$

Make sure you can use these expressions to approximate the vector potential in the various regimes considered in this week's activity.

Recall further that the derivative (gradient) of the scalar potential gives (minus) the electric field, whose derivative (divergence) in turn yields the charge density (divided by ϵ_0). We expect an analogous construction starting with the vector potential.

The only derivative we have so far for vector fields is the divergence, so we investigate $\vec{\nabla}\cdot\vec{\boldsymbol{A}}$. Consider the current ring in this week's activity. We know that $\vec{\boldsymbol{A}}$ takes the form

$$\vec{\boldsymbol{A}} = A_\phi(r,z)\,\hat{\boldsymbol{\phi}}$$

What is the flux of $\vec{\boldsymbol{A}}$ out of a small "pineapple"-shaped box? The only surfaces which contribute are those on which ϕ = constant, and since A_ϕ doesn't depend on ϕ the flux in one side must equal the flux out the other. We conclude that

$$\vec{\nabla}\cdot\vec{\boldsymbol{A}} = 0$$

(This is actually a choice: There is considerable freedom in the definition of the vector potential, analogous to adding a constant to the scalar potential. The expression we are using for the vector potential leads to the *gauge condition* that the divergence of $\vec{\boldsymbol{A}}$ vanish. Put differently, there is no physics in $\vec{\nabla}\cdot\vec{\boldsymbol{A}}$, so we can safely set it to zero.)

So the divergence is not the right kind of derivative here. Actually, we knew that already, since the magnetic field is a vector field, and the divergence is a scalar. Flux doesn't work; what other integrals could we try? Instead of the change in the normal component of $\vec{\boldsymbol{A}}$, let's measure the tangential component. We therefore consider the *circulation* $\oint \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{r}}$ of $\vec{\boldsymbol{A}}$ around a closed loop, which leads us to another derivative, namely the curl.

We have seen that the vector potential for a uniform ring of current takes the form

$$\vec{\boldsymbol{A}} = A_\phi(r,z)\,\hat{\boldsymbol{\phi}}$$

It is easy to argue that the circulation of $\vec{\boldsymbol{A}}$ around a loop in the rz-plane is zero, since $\hat{\boldsymbol{\phi}}$ is perpendicular to the loop. Thus, the $\hat{\boldsymbol{\phi}}$-component of $\vec{\nabla} \times \vec{\boldsymbol{A}}$ must vanish. Consider now a loop in the ϕz-"plane". The vertical sides again don't contribute, and $\vec{\boldsymbol{A}}$ is strongest near $z = 0$. Using the right-hand rule (and assuming positive charge density λ), this means that the $\hat{\boldsymbol{r}}$-component of $\vec{\nabla} \times \vec{\boldsymbol{A}}$ must be positive for $z > 0$, negative for $z < 0$, and zero for $z = 0$. (This argument fails along the z-axis, where however $\vec{\nabla} \times \vec{\boldsymbol{A}}$ must point in the $\hat{\boldsymbol{z}}$ direction by symmetry.)

What about the $\hat{\boldsymbol{z}}$-component? Consider first a small loop around the axis. Since $\vec{\boldsymbol{A}}$ is proportional to $\hat{\boldsymbol{\phi}}$, the circulation around such a loop is clearly positive. Thus, the $\hat{\boldsymbol{z}}$-component of $\vec{\nabla} \times \vec{\boldsymbol{A}}$ must be positive everywhere along the axis.

How about away from the axis? Examine a loop in the xy-plane, whose sides are $r =$ constant and $\phi =$ constant. The radial sides don't contribute, since $\hat{\boldsymbol{\phi}}$ is perpendicular to them. The inner and outer sides will contribute, and $\vec{\boldsymbol{A}}$ will be stronger on the inner side, which is shorter. So it's not easy to tell what the z-component of $\vec{\nabla} \times \vec{\boldsymbol{A}}$ will be.

The vector potential due to a ring of current, in the plane of the ring, with either $r \ll R$ or $r \gg R$, turns out to be

$$\vec{\boldsymbol{A}} = \frac{\mu_0 I}{4} \begin{cases} \frac{r}{R}\left(1 + \frac{3}{8}\frac{r^2}{R^2} + \ldots\right)\hat{\boldsymbol{\phi}} & (r \ll R) \\ \frac{R^2}{r^2}\left(1 + \frac{3}{8}\frac{R^2}{r^2} + \ldots\right)\hat{\boldsymbol{\phi}} & (r \gg R) \end{cases}$$

Thus, $\vec{\boldsymbol{A}}$ falls off like $\frac{1}{r^2}$ as r goes to infinity, while the length of the circular arcs increases like r. Far enough out, the big side loses. Again invoking the right-hand rule, we see that the z-component of $\vec{\nabla} \times \vec{\boldsymbol{A}}$ is negative, for r large and $z = 0$.

Putting all of this together, it looks like $\vec{\nabla} \times \vec{\boldsymbol{A}}$ loops around the current, in the direction determined by the right-hand rule. This is what we expect for the magnetic field of a current-carrying wire, and it should not surprise you to learn that we can in fact *define* the magnetic field $\vec{\boldsymbol{B}}$ in general via

$$\vec{\boldsymbol{B}} = \vec{\nabla} \times \vec{\boldsymbol{A}}$$