PH 422: Day 3

Please read Sections 3.8, 3.9, 3.10 from the mathematics notes.

22 Differential form of Gauss' Law

Recall that Gauss' Law says that

$$\int_{\text{box}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

But the enclosed charge is just

$$Q_{\text{inside}} = \int_{\text{box}} \rho \, dV$$

so we have

$$\int_{\text{box}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{1}{\epsilon_0} \int_{\text{box}} \rho \, dV$$

Putting this all together, the Divergence Theorem tells us that

$$\int_{\text{inside}} \vec{\nabla} \cdot \vec{\boldsymbol{E}} \; dV = \frac{1}{\epsilon_0} \int_{\text{box}} \rho \, dV$$

for *any* closed box. This means that the integrands themselves must be equal, that is,

$$\vec{\nabla} \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0}$$

This is the differential form of Gauss' Law, and is one of Maxwell's Equations. It states that the divergence of the electric field at any point is just a measure of the charge density there.

23 The Divergence of a Coulomb Field

The electric field of a point charge at the origin is given by

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \frac{q \, \hat{\boldsymbol{r}}}{r^2}$$

We can take the divergence of this field using the expression derived above for the divergence of a radial vector field, which yields

$$\vec{\nabla} \cdot \vec{\boldsymbol{E}} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 E_r \right) = \frac{1}{4\pi\epsilon_0} \frac{q\,\hat{\boldsymbol{r}}}{r^2} \frac{1}{r^2} \frac{\partial q}{\partial r} = 0$$

On the other hand, the flux of this electric field through a sphere centered at the origin is

$$\int\limits_{\text{sphere}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \int\limits_{\text{sphere}} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\, dA = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left(4\pi r^2 \right) = \frac{q}{\epsilon_0}$$

in agreement with Gauss' Law. The Divergence Theorem then tells us that

$$\int\limits_{\text{inside}} \vec{\nabla} \cdot \vec{\boldsymbol{E}}\, dV = \int\limits_{\text{sphere}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} \neq 0$$

even though $\vec{\nabla} \cdot \vec{\boldsymbol{E}} = 0$!. What's going on?

A bit of thought yields a clue: $\vec{\boldsymbol{E}}$ isn't defined at $r = 0$; neither is its divergence. So we have a function which vanishes almost everywhere, whose integral isn't zero. This should remind you of the Dirac delta function. However, we're in 3 dimensions here, so that the correct conclusion is

$$\vec{\nabla} \cdot \frac{1}{4\pi\epsilon_0} \frac{q\,\hat{\boldsymbol{r}}}{r^2} = \frac{q}{\epsilon_0}\, \delta^3(\vec{\boldsymbol{r}}) = \frac{q}{\epsilon_0}\, \delta(x)\, \delta(y)\, \delta(z)$$

or equivalently

$$\rho = q\, \delta^3(\vec{\boldsymbol{r}})$$

which should not be surprising.

24 Electric Field Lines

You probably already know the following facts about electric field lines:

- The density of field lines is proportional to the strength of the electric field there;
- Field lines only start at positive charges and end at negative charges;
- Field lines never cross.

These rules can all be explained using Gauss' Law, since the flux of the electric field can be interpreted as counting the number of field lines which cross the surface.