# PH 422: Day 2

## 19 Gauss' Law

Recall Gauss' Law, which says that

$$\oint_{\text{box}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

Consider the example of a point charge inside an imaginary sphere of radius $r$. The electric field due to the charge is

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \,\hat{\boldsymbol{r}}$$

and the surface element is

$$d\vec{\boldsymbol{A}} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{\boldsymbol{r}}$$

Thus, the flux is given by

$$\begin{aligned} \int\limits_{\text{sphere}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} &= \int\limits_{\text{sphere}} \frac{1}{4\pi\epsilon_0} \frac{q\,\hat{\boldsymbol{r}}}{r^2} \cdot \hat{\boldsymbol{r}} \, dA \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \times (\text{surface area}) \\ &= \frac{q}{\epsilon_0} \end{aligned}$$

as claimed. Note that the value of this integral does not depend on the radius $r$ of the spherical box.

## 20 Example: Flux through a cube

The integral in Gauss' Law does not depend on the shape of the surface being used. So let's replace the sphere in the previous example with a cube. Suppose the charge is at the origin, and the length of each side of the cube is 2. What is the flux through one face? We have

$$\text{flux} = \int\limits_{-1}^{1} \int\limits_{-1}^{1} \frac{1}{4\pi\epsilon_0} \frac{q\,\hat{\boldsymbol{r}}}{r^2} \cdot \hat{\boldsymbol{k}} \, dx \, dy = \frac{q}{4\pi\epsilon_0} \int\limits_{-1}^{1} \int\limits_{-1}^{1} \frac{dx \, dy}{\sqrt{x^2 + y^2 + 1}}$$

This integral can be computed with the help of integral tables. Alternatively, the worksheet `flux.mws` can be used to compute the answer. Does the result agree with the previous computation for the sphere? Now suppose the charge is not at the origin. Maple can still do the integrals numerically; try some examples. Finally, suppose the charge is on a face or an edge of the cube. What answer do you expect? Check and see.

## 21 Gauss's Law and Symmetry

Recall that the electric field of a uniform disk is given along the axis by

$$\vec{\boldsymbol{E}}(z) = \frac{2\pi\sigma}{4\pi\epsilon_0}\left(\frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{\boldsymbol{z}}$$

where of course $\frac{z}{\sqrt{z^2}} = \pm 1$ depending on the sign of $z$. (The notation $\mathrm{sgn}(z)$ is often used to represent the sign of $z$, in order to simplify expressions like $\frac{z}{\sqrt{z^2}}$.) In the limit as $R \to \infty$, one gets the electric field of a uniformly charged plane, which is just

$$\vec{\boldsymbol{E}}(z) = \mathrm{sgn}(z)\,\frac{\sigma}{2\epsilon_0}\,\hat{\boldsymbol{z}}$$

which is valid everywhere, as any point can be thought of as being on the axis. But the calculation leading to the first expression above was somewhat involved. Is there a better way?

An infinite plane of charge is highly symmetric. Not only is every point like every other, at any point on the plane, all directions in the plane are equivalent. This rotational symmetry means that, at all points above the plane, the electric field must be orthogonal to the plane. How do we prove this statement? With proof by contradiction: Suppose that, at a particular point above the plane, the electric field had a component parallel to the plane. Let's call the direction of the electric field the $x$-direction. An observer, standing at this point, might initially face so as to see the electric field pointing straight in front of him. But, if that observer then turns in place to face a different direction, he would expect to see the electric field pointing straight in front of him because of the symmetry of what he sees. Indeed, if he had been blindfolded when he turned, he could not tell, by looking at the charge distribution everywhere, that he had turned. Nevertheless, the fact that the observer has turned has not changed the electric field. If it initially had a component in the $x$-direction, it still must have a component in the $x$-direction. The only way to resolve this contradiction is to recognize that our original assumption cannot be true. The electric field cannot have a component parallel to the plane. A similar argument using the translational symmetry shows that the electric field can only depend on the distance from the plane. And another similar argument shows that the electric field must in fact point in opposite directions on opposite sides of the plane. The electric field must therefore be of the form

$$\vec{\boldsymbol{E}} = E(z)\,\hat{\boldsymbol{z}}$$

assuming that the $z$-direction is orthogonal to the plane.

Now recall Gauss' Law, which relates the flux of the electric field through any closed surface to the charge enclosed by the surface, that is,

$$\int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{q}{\epsilon_0}$$

Choose a closed surface which exploits the symmetry. A rectangular box is one possibility, two of whose faces are parallel to the plane, and equidistant from it. The flux through the sides of this box is zero, since the normal vector to these sides is parallel to the plane, but $\vec{\boldsymbol{E}}$ is perpendicular to the plane, so that $\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = 0$. What about the top? The electric field is perpendicular to the top, but constant in magnitude. Thus,

$$\int_{\text{top}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = E(z)A$$

where $z$ is the distance from the plane to the top of the box (where $z > 0$). By symmetry, the flux through the bottom of the box is the same as that though the top. This implies that $E(-z) = -E(z)$. Finally, the charge enclosed by the box is just the charge density $\sigma$ times the area of the part of the plane inside the box, which is again $A$. Inserting all of this into Gauss' Law, we obtain

$$2E(z)A = \frac{\sigma A}{\epsilon_0}$$

so that

$$E(z) = \frac{\sigma}{2\epsilon_0}$$

which turns out to be independent of $z$ (for $z > 0$).