$$\begin{tikzcd}
V \arrow[d, bend left, "5"'] \arrow[dd, bend left=60, "6"] \\
\vec{\boldsymbol{E}} \arrow[u, bend left, "3"'] \arrow[d, bend left, "4"'] \\
\rho \arrow[u, bend left, "2"'] \arrow[uu, bend left=60, "1"]
\end{tikzcd}$$

Figure 5.5: The relationships between V, $\vec{\boldsymbol{E}}$, and ρ. Each numbered arrow is discussed in the text; moving down the diagram corresponds to differentiation.

PH 422: Day 16

Please read Sections 3.16–3.15 from the mathematics notes.

38 Conductors

Please read Sections ?? in Griffiths.

39 The Relationship between $\vec{\boldsymbol{E}}$, V, and ρ

Starting with the electric potential for a point charge, we used the superposition principle to obtain an integral formula for the potential due to any charge distribution, as well as a similar formula for the electric field. These relationships correspond, respectively, to the arrows numbered 1 and 2 in Figure 5.5. Furthermore, the electric field is just (minus) the gradient of the potential, which can be inverted to obtain the potential as an integral of the electric field; these are arrows 5 and 3, respectively. We have also seen that the charge density can be recovered as the divergence of the electric field, which is arrow number 4.

It remains to show how to recover the charge density from the potential (arrow number 6), at which point we are able to compute any of $\vec{\boldsymbol{E}}$, V, and ρ from any other.

But we know that

$$\begin{aligned} \vec{\boldsymbol{E}} &= -\vec{\nabla} V \\ \frac{\rho}{\epsilon_0} &= \vec{\nabla} \cdot \vec{\boldsymbol{E}} \end{aligned}$$

from which it is easy to compute

$$\frac{\rho}{\epsilon_0} = -\vec{\nabla} \cdot \vec{\nabla} V$$

which is the desired relation; this is *Poisson's equation.* These relationships are nicely summarized in the triangle in Griffiths in Figure 2.35 on page 87, or alternatively in Figure 5.5.

The special case of Poisson's equation with no source, namely

$$\triangle V = 0$$

is called *Laplace's equation.* This equation arises in many contexts, not just in electrodynamics.

40 The Relationship between $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{A}}$, and $\vec{\boldsymbol{J}}$

Starting with the vector potential for a line current, we used the superposition principle to obtain an integral formula for the vector potential due to any current distribution, as well as a similar formula, the Biot-Savart Law, for the magnetic field. These relationships correspond, respectively, to the arrows numbered 1 and 2 in Figure 5.6. Furthermore, the magnetic field is just the curl of the vector potential, which can (in principle) be inverted to obtain the vector potential as an integral of the magnetic field; these are arrows 5 and 3, respectively.

We have also seen that the current density can be recovered as the curl of the magnetic field, which is arrow number 4. It remains to show how to recover the current density from the vector potential (arrow number 6), at which point we are able to compute any of $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{A}}$, and $\vec{\boldsymbol{J}}$ from any other. But we know that

$$\begin{aligned} \vec{\boldsymbol{B}} &= \vec{\nabla} \times \vec{\boldsymbol{A}} \\ \mu_0 \vec{\boldsymbol{J}} &= \vec{\nabla} \times \vec{\boldsymbol{B}} \end{aligned}$$

$$\begin{tikzcd}
\vec{\boldsymbol{A}} \arrow[d, bend left, "5"'] \arrow[dd, bend left=60, "6"] \\
\vec{\boldsymbol{B}} \arrow[u, bend left, "3"'] \arrow[d, bend left, "4"'] \\
\vec{\boldsymbol{J}} \arrow[u, bend left, "2"'] \arrow[uu, bend left=60, "1"]
\end{tikzcd}$$

Figure 5.6: The relationships between $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{J}}$. Each numbered arrow is discussed in the text; moving down the diagram corresponds to differentiation. Compare Figure 5.5.

from which it is easy to compute

$$\mu_0 \vec{\boldsymbol{J}} \;=\; \vec{\nabla} \times (\vec{\nabla} \times \vec{\boldsymbol{A}})$$

which is the desired relation. These relationships are nicely summarized in the triangle in Griffiths in Figure 5.48 on page 240, or alternatively in Figure 5.6. (This vector second derivative can be rewritten using formulas on the inside front cover of Griffiths.) The corresponding relationships between $\vec{\boldsymbol{E}}$, V, and ρ are summarized in the triangle in Griffiths in Figure 2.35 on page 87, and in Figure 5.5.

41 Second derivatives and Maxwell's Equations

Since the electric field is (minus) the gradient of the scalar potential, it is conservative. But since

$$\vec{\nabla} \times \vec{\nabla} V = \vec{\mathbf{0}}$$

for *any* function V, we can rewrite this as

$$\vec{\nabla} \times \vec{\boldsymbol{E}} = \vec{\mathbf{0}}$$

which is the differential form of (the electrostatic version of) Faraday's Law, and another of Maxwell's Equations.

Similarly, we can apply the identity

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\boldsymbol{F}}) = 0$$

for *any* vector field $\vec{\boldsymbol{F}}$, to the magnetic vector potential, which yields the fourth and final of Maxwell's Equations, namely

$$\vec{\nabla} \cdot \vec{\boldsymbol{B}} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{\boldsymbol{A}}) = 0$$

In summary, Maxwell's equations *for electro- and magnetostatics* are:

$$\begin{aligned} \vec{\nabla} \cdot \vec{\boldsymbol{E}} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{\boldsymbol{E}} &= \vec{\boldsymbol{0}} \\ \vec{\nabla} \cdot \vec{\boldsymbol{B}} &= 0 \\ \vec{\nabla} \times \vec{\boldsymbol{B}} &= \mu_0 \, \vec{\boldsymbol{J}} \end{aligned}$$