PH 422: Day 11

Please read Sections 5.1–5.8 from the mathematics notes.

32 The magnetic field of a uniform planar current

Consider the magnetic field due to a uniform current everywhere in the xy-plane, moving in the x-direction. In what direction is the magnetic field? The Biot-Savart Law says that

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\int \frac{\vec{\boldsymbol{K}}(\vec{\boldsymbol{r}}') \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')\, dA'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

But $\vec{\boldsymbol{K}} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')$ is perpendicular to $\vec{\boldsymbol{K}}$, and since $\vec{\boldsymbol{K}}$ points (everywhere) in the x-direction, the $\hat{\boldsymbol{\imath}}$-component of $\vec{\boldsymbol{B}}$ must vanish, that is, $B_x = 0$.

Consider now the z-component, B_z. The Biot-Savart Law implies that reversing the direction of $\vec{\boldsymbol{K}}$ (everywhere) must also reverse the direction of $\vec{\boldsymbol{B}}$. Suppose that $B_z > 0$ somewhere. Stand there, and turn around. On the one hand, the magnetic field can't change — you've moved, but it hasn't. Thus, B_z must still be positive. On the other hand, what you now see is exactly the same as if the xy-plane had turned underneath you, reversing the direction of $\vec{\boldsymbol{B}}$. By this argument, B_z must now be negative! The only way out of this predicament is for the z-component of $\vec{\boldsymbol{B}}$ to vanish, that is, $B_z = 0$.

Furthermore, $\vec{\boldsymbol{B}}$ clearly doesn't depend on x or y — the plane looks the same everywhere. Putting this all together, we conclude that

$$\vec{\boldsymbol{B}} = B_y(z)\,\hat{\boldsymbol{\jmath}}$$

Recall Ampère's Law, which relates the circulation of the magnetic field around any closed loop to the current enclosed by the loop, that is,

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \mu_0 I$$

We can use this to find $\vec{\boldsymbol{B}}$ by choosing an appropriate loop. Since $\vec{\boldsymbol{B}}$ points in the y-direction, choose a rectangular loop, two of whose sides are parallel to the y-axis. Since B_y only depends on z, these two sides should be at different z values. In other words, choose a loop in the yz-plane. Evaluate the LHS of Ampère's Law. The two sides parallel to the z-axis don't contribute, and each of the other two sides contributes $B_y(z)$ times the length of the side.

Assuming the loop does not intersect the xy-plane, the RHS vanishes; this forces B_y to be constant. Well, not quite: this forces B_y to be constant on each side of the xy-plane, but doesn't show that it's the same constant on both sides. In fact, there is another symmetry argument which says this can't be the case. Take the xy-plane, and rotate it 180° about the x-axis. This doesn't change the current, but it takes, for instance, $\hat{\boldsymbol{\jmath}}$ above the plane to $-\hat{\boldsymbol{\jmath}}$ below.

So consider a loop which does intersect the xy-plane. Now the contributions on the two sides parallel to the y-axis will add. Orient the loop in the $-\hat{\boldsymbol{\jmath}}$ direction for $z > 0$; this is the correct "right-hand rule" orientation for Ampère's Law. Let B_y denote the (constant) magnitude of the magnetic field for $z > 0$. Then the LHS of Ampère's Law yields $-2B_y$, times the length of the loop in the y direction. But the RHS is the current flowing through the loop, which is

$$I = \int \vec{\boldsymbol{K}} \cdot \hat{\boldsymbol{n}}\, ds$$

where the integral is taken along the intersection of the loop with the plane, and where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}}$. The current is therefore $|\vec{\boldsymbol{K}}|$ times the length, and we conclude that $-2B_y = |\vec{\boldsymbol{K}}|$, or equivalently

$$\vec{\boldsymbol{B}} = -\frac{\mu_0}{2}\,|\vec{\boldsymbol{K}}|\,\hat{\boldsymbol{\jmath}} \qquad (z > 0)$$

and of course B_y has the opposite sign for $z < 0$.

33 Ampère's Law and Symmetry