Time evolution of the system's state

Let's say that at \( t = t_0 \) our system is in a state \( |\psi_0\rangle \). At what state is it going to be at some later time \( t \)? \( \Rightarrow \) i.e. we need to find \( |\psi(t)\rangle \), 

\[
1_{d, t_0; t} \rightarrow t >, t > t_0
\]

\[
\lim_{t \rightarrow t_0} |\psi_{1, t_0; t} \rangle = |\psi_{1, t} \rangle
\]

Notation: \( |\psi_{1, t_0; t} \rangle \equiv |\psi_{1, t} \rangle \)

So, how does the transformation \( |\psi_{1, t_0; t} \rangle \Rightarrow |\psi_{1, t} \rangle \) happen? \( \Rightarrow \)

\[
|\psi_{1, t_0; t} \rangle = \hat{U}(t, t_0) |\psi_{1, t_0} \rangle
\]

\[
\hat{U} + \hat{U} = \hat{I}
\]

\[
\hat{U}(t_0, t_0) = \hat{I}
\]

Unitary \( \Rightarrow \) time-evolution operator

\[
\frac{\partial}{\partial t}
\]

propagator
Recall from Lecture #11 that the generator of time translations is the Hamiltonian \( \hat{H} \) so that
\[
\hat{U}(t_0 + dt, t_0) = \mathbf{1} - \frac{i}{\hbar} \hat{H} dt.
\]
Consider successive transformations: first from \( t_0 \) to \( t \), described by \( \hat{U}(t, t_0) \), and then from \( t \) to \( t + dt \), described by \( \hat{U}(t + dt, t) \) so that
\[
\hat{U}(t + dt, t) \hat{U}(t, t_0) = \hat{U}(t + dt, t_0) = (\mathbf{1} - \frac{i}{\hbar} \hat{H} dt) \hat{U}(t, t_0) = \hat{U}(t, t_0) - \frac{i}{\hbar} dt \hat{H} \hat{U}(t, t_0)
\]
\[
\hat{U}(t + dt, t_0) - \hat{U}(t, t_0) = -\frac{i}{\hbar} dt \hat{H} \hat{U}(t, t_0)
\]
\[
\frac{\partial}{\partial t} \hat{U}(t, t_0) \cdot dt = -i \hbar \frac{\partial}{\partial t} \hat{U}(t, t_0) = \hat{H} \hat{U}(t, t_0) 
\]
the Schrödinger equation for the propagator.
\[ \text{Multiply by } |x, t_0 \rangle \Rightarrow \]
\[
i \hbar \frac{\partial}{\partial t} \hat{U}(t,t_0) |x, t_0 \rangle = \hat{H} \hat{U}(t,t_0) |x, t_0 \rangle \]
\[
|x, t_0 \rangle \quad |x, t_0 \rangle \]

\[
i \hbar \frac{\partial}{\partial t} |x, t_0, t \rangle = \hat{H} |x, t_0, t \rangle \Rightarrow \]

Go back to \( \hat{U}(t, t_0) \) \( \Rightarrow \) Schroedinger equation!

can we integrate Eq. (15.1) and find \( \hat{U}(t, t_0) \)? \( \Rightarrow \) sure!

Case 1 \( \hat{H} \neq \hat{H}(t) \) (as in most "textbook problems")

Then \( \Rightarrow \)
\[
\hat{U}(t, t_0) = e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \quad (15.2)
\]

Case 2 \( \hat{H} = \hat{H}(t) \), but \( \left[ \hat{H}(t_1), \hat{H}(t_2) \right] = 0 \)

\[
\forall \text{ for any } t_1, t_2
\]
\[
\hat{U}(t_1, t_0) = e^{-\frac{i}{\hbar} \int_{t_0}^{t_1} dt' \hat{H}(t')} \quad (15.3)
\]

This obviously turns into (15.2) if \( \hat{H} \neq \hat{H}(t') \)
Case 3 \( [\hat{A}(t_1), \hat{A}(t_2)] \neq 0 \) for arbitrary \( t_1, t_2 \)

\[
\hat{U}(t, t_0) = 1 + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \int_{t_0}^{t} \cdots \int_{t_0}^{t_1} \hat{A}(t_{n-1}) \hat{A}(t_n) dt_{n-1} \cdots dt_1
\]

(15.4)

\( \uparrow \) \text{Dyson series} \( \uparrow \)

used \( \uparrow \) in time-dependent perturbation theory

(Phys 653)

Homework:

Start from Eq. (15.4), make the assumption of \( [\hat{A}(t_1), \hat{A}(t_2)] = 0 \) and arrive at Eq. (15.3)

How do we propagate an arbitrary state \( |\psi(t)\rangle \) in time? \( \Rightarrow \) Consider a basis \( \{ |\psi_n\rangle \} \) of eigenvectors of an operator \( \hat{A} \), so that \( \hat{A} |\psi_n\rangle = a_n |\psi_n\rangle \)

If \( [\hat{A}, \hat{A}] = 0 \) \( \Rightarrow \hat{A} \) and \( \hat{A} \) share the same basis \( \Rightarrow \) Hamiltonian, consider \( \hat{A} = \hat{H}(t) \)
\[ |\psi_n\rangle = E_n |\psi_n\rangle \]

As we know from before, we can expand an arbitrary state vector \( |\psi\rangle \) in terms of the base vectors \( |\psi_n\rangle \) as follows:

\[ |\psi\rangle = \sum_n c_n(t) |\psi_n\rangle \] suppose this is the initial state at time \( t_0 = 0 \),

i.e. \( |\psi\rangle \equiv |\psi, t_0 = 0\rangle \).

What is \( |\psi, t_0 = 0; t\rangle \)?

\[ |\psi, t_0 = 0; t\rangle = \hat{U}(t, t_0 = 0) |\psi, t_0 = 0\rangle = e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} |\psi, t_0 = 0\rangle = e^{-\frac{i}{\hbar} \hat{H}t} \sum_n c_n(t_0) |\psi_n\rangle = \sum_n c_n(0) e^{-\frac{i}{\hbar} E_n t} |\psi_n\rangle \]

\[ \uparrow \]

call: if \( \hat{A}|\psi_n\rangle = E_n |\psi_n\rangle \)

\[ f(\hat{A}) |\psi_n\rangle = f(E_n) |\psi_n\rangle \]
What if at $t_0 = 0$, we are in one of the eigenstates $|\psi_k\rangle$? Then $|\ldots, t_0 = 0\rangle = |\psi_k\rangle$.

$|\ldots, t_0 = 0; t\rangle = C_k(t) |\psi_k\rangle = C_k(0) \cdot e^{-\frac{i}{\hbar}E_k t} |\psi_k\rangle = e^{-\frac{i}{\hbar}E_k t} |\psi_k\rangle$

We are still at the same physical state $|\psi_k\rangle$, but acquired the phase modulation $e^{-\frac{i}{\hbar}E_k t}$.

So, general procedure to find time-evolution of $|\ldots\rangle$:

1) Find $\hat{A}$ such that $[\hat{A}, \hat{H}] = 0$
2) Expand $|\ldots\rangle$ in terms of $|\psi_n\rangle$
3) Change $C_n(0)$ to $C_n(0) e^{-\frac{i}{\hbar}E_n t}$

Example

Consider a system whose initial state $|\psi(0)\rangle$ is $|\psi(0)\rangle = |\ldots, t_0 = 0\rangle = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

Hamiltonian is represented by

$\hat{H} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix}$

Find the state of the system at a later time $t$. 


1) Find eigenvectors of $A$ \[\begin{bmatrix} 3 - \lambda & 0 & 0 \\ 0 & -1 & 5 \\ 0 & 5 & -1 \end{bmatrix} = 0 \Rightarrow\]

\[\lambda_1 = 3, \quad \lambda_{2,3} = \pm 5\]

$|\Psi_1\rangle = |\lambda = 3\rangle \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 5 \\ 0 & 5 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_5 \end{bmatrix} = 0$

$c_1$ - arbitrary

\[\Rightarrow -3c_2 + 5c_3 = 0 \Rightarrow c_2 = -\frac{5}{3}c_3\]

$\Psi_2 \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$c_1 = 0, \quad c_2 = c_3$

$|\Psi_2\rangle = |\lambda = -5\rangle \Rightarrow \begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_5 \end{bmatrix} = 0 \Rightarrow$

$c_1 = 0, \quad c_2 = -c_5$
2) Now expand $|\Psi(0)\rangle$ in terms of coefficients of expansion: $|\Psi_1\rangle$, $|\Psi_2\rangle$, $|\Psi_3\rangle$.

$$\langle \Psi_1 | \Psi(0) \rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{4}} \\ \frac{0}{\sqrt{4}} \end{bmatrix} = \frac{3}{\sqrt{4}} = \frac{3}{2}$$

$$\langle \Psi_2 | \Psi(0) \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{4}} \\ \frac{0}{\sqrt{4}} \end{bmatrix} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\langle \Psi_3 | \Psi(0) \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{4}} \\ \frac{0}{\sqrt{4}} \end{bmatrix} = -\frac{1}{\sqrt{2}}$$

So,

$$|\Psi(0)\rangle = \frac{3}{5} |\Psi_1\rangle + \frac{1}{\sqrt{2}} |\Psi_2\rangle - \frac{1}{\sqrt{2}} |\Psi_3\rangle.$$  

3) Now propagate $|\Psi(0)\rangle$ in time:

$$|\Psi(t)\rangle = |\Psi(t)\rangle_{t = 0} = \frac{3}{5} e^{-\frac{1}{16} \cdot 5t} |\Psi_1\rangle + \frac{1}{\sqrt{2}} e^{-\frac{1}{16} \cdot 5t} |\Psi_2\rangle - \frac{1}{\sqrt{2}} e^{\frac{1}{16} \cdot 5t} |\Psi_3\rangle =$$

$$= \frac{1}{5} \begin{bmatrix} 3 e^{-\frac{1}{16} \cdot 5t} \\ \frac{4}{\sqrt{2}} e^{-\frac{1}{16} \cdot 5t} \\ -\frac{4}{\sqrt{2}} e^{\frac{1}{16} \cdot 5t} \end{bmatrix}$$