Homework #6
(due Wednesday, November 15, 2017)

1. (10 pts) Show that $[\mathbf{R} \cdot \mathbf{P}, H] = 2 \mathbf{i} \hbar T - \mathbf{i} \hbar \mathbf{R} \cdot \nabla V$, where $\mathbf{R}$ is the position operator in 3D space, $\mathbf{P}$ is the momentum operator, $H$ is the Hamiltonian ($H = \mathbf{P}^2/2m + V(\mathbf{R})$), and $T$ is the kinetic energy operator ($T = \mathbf{P}^2/2m$).

2. (20 pts) A particle of mass $m$, which moves inside an infinite 1D potential well of length $a$, is described by the following wave function at $t = 0$:

$$\psi(x,0) = \frac{A}{\sqrt{a}} \sin \left( \frac{\pi x}{a} \right) + \sqrt{\frac{3}{5a}} \sin \left( \frac{3\pi x}{a} \right) + \sqrt{\frac{1}{5a}} \sin \left( \frac{5\pi x}{a} \right),$$

where $A$ is a real constant.

(a) Find $A$ so that $\psi(x,0)$ is normalized

(b) If measurements of the energy are carried out at $t=0$, what are the values that will be found and what are the corresponding probabilities?

(c) What is the average energy?

(d) Find the wave function $\psi(x,t)$ at a later time $t$

(e) Determine the probability of finding the system at a time $t$ in the state

$$\varphi(x,t) = \sqrt{\frac{2}{a}} \sin \left( \frac{5\pi x}{a} \right) e^{-iE_5 t/\hbar}$$

(f) The same as (e) but for the state $\chi(x,t) = \sqrt{\frac{2}{a}} \sin \left( \frac{2\pi x}{a} \right) e^{-iE_2 t/\hbar}$

3. (20 pts) Sakurai 2.9.

4. (10 pts) Consider a particle of mass $m$ in the 1D potential well given by

$V(x) = 0$ if $|x| < a$ and $V(x) = +\infty$ if $|x| > a$. Let’s say that the particle is in the ground state (i.e. in the state with the lowest energy). Suddenly, the well symmetrically expands to twice its size. What is the probability to find the particle in the ground state of this new well?
5. (30 pts) Consider a wave packet freely moving in 1D so that the wave function at 
\( t = 0 \) is given by

\[
\psi(x,0) = A \exp \left[ -\frac{x^2}{2a^2} + i \frac{p_0}{\hbar} x \right],
\]

where \( p_0 \) is a momentum of the particle, and \( A \) is the normalization constant.

(a) What is the probability to find the particle in the region \([-\Delta, \Delta]\), where \( \Delta \) is a 
very small parameter ?

(b) What is the uncertainty of the measurement of \( x \) in this state ?

(c) Now consider the state of this system at some later time \( t \) and find \( \psi(x,t) \) and 
the probability density \( |\psi(x,t)|^2 \).

**Hint:** the easiest way is to expand \( \psi(x,0) \) in terms of the momentum eigenstates 
and then propagate them in time.

Make sure to check your function \( \psi(x,t) \) (that at \( t = 0 \) you get the initially given 
\( \psi(x,0) \)).

Don’t be afraid of a very long expression you obtained in (c) – just rearrange the 
terms in a way that you can actually analyze the function in order to answer the 
following questions:

(d) Did the probability to find the particle in the region \([-\Delta, \Delta]\) change ? If yes, 
how (a qualitative answer is fine) ?

(e) Did the uncertainty of the measurement of \( x \) change ? If yes, how (a 
qualitative answer is fine) ?

6. Reading assignment: Sakurai 2.1-2.2, 2.4.