The Propagation of Electromagnetic Waves

PROBLEM

A surface $H$ wave can be propagated along a plane boundary between two media whose permittivities $\varepsilon_1$ and $\varepsilon_2$ are of opposite signs. The wave is damped in both media. Determine the relation between the frequency and the wave number.

SOLUTION. We take the boundary surface as the $xy$-plane, the wave being propagated in the $x$-direction and the field $H$ being in the $y$-direction. Let the half-space $z > 0$ contain the medium with the positive permittivity $\varepsilon_1$, and the half-space $z < 0$ that with the negative permittivity $\varepsilon_2$. We seek the field in the wave damped as $z \to \pm \infty$ in the form

$$H_1 = H_0 e^{ik_1 x - \kappa_1 z}, \quad \kappa_1 = \sqrt{k^2 - \omega^2 \varepsilon_1 / c^2} \quad \text{for } z > 0,$$

$$H_2 = H_0 e^{ik_2 x + \kappa_2 z}, \quad \kappa_2 = \sqrt{k^2 + \omega^2 \varepsilon_2 / c^2} \quad \text{for } z < 0,$$

where $k$, $\kappa_1$, and $\kappa_2$ are real. The boundary condition that $H_1 = H_2$ be continuous is already satisfied, and the continuity condition on $E_y$ gives $(1/\varepsilon_1) \partial H_1 / \partial z = (1/\varepsilon_2) \partial H_2 / \partial z$ for $z = 0$, or $\kappa_1 / \varepsilon_1 = \kappa_2 / \varepsilon_2$. This equation can be satisfied if $\varepsilon_1 < \varepsilon_2$ (and if $\varepsilon_1 \varepsilon_2 < 0$, as has been assumed). The relation between $k$ and $\omega$ is

$$k^2 = \omega^2 \varepsilon_1 |\varepsilon_2| / c^2 (|\varepsilon_1| - \varepsilon_2).$$

It is easily seen that surface $E$ waves cannot be propagated.

§89. The reciprocity principle

The emission of monochromatic electromagnetic waves from a source consisting of a thin wire in an arbitrary medium is described by the equations

$$\text{curl } E = i \omega B / c, \quad \text{curl } H = -i \omega D / c + 4 \pi \mathbf{j}_x / c,$$

(89.1)

where $\mathbf{j}_x$ is the density of periodic currents flowing in the wire which are extraneous to the medium.

Let two different sources (of the same frequency) be placed in the medium; we denote by the suffixes 1 and 2 the fields due to these sources separately. The medium may be inhomogeneous and anisotropic. The only assumption which we shall make concerning the properties of the medium is that the linear relations $D_i = \varepsilon_{ik} E_k$, $B_i = \mu_{ik} H_k$ hold, the tensors $\varepsilon_{ik}$ and $\mu_{ik}$ being symmetrical. Under these conditions it is possible to derive a relation between the fields of the two sources and the extraneous currents in them.

We take the scalar products of the two equations $\text{curl } E_1 = ik B_1$, $\text{curl } H_1 = -ik D_1 + 4 \pi \mathbf{j}_{x,1} / c$ with $H_2$ and $H_2$ respectively, and of the corresponding equations for $E_2$ and $H_2$ with $-H_1$ and $-E_1$. Adding all four together, we obtain

$$(H_2 \cdot \text{curl } E_1 - E_1 \cdot \text{curl } H_2) + (E_2 \cdot \text{curl } H_1 - H_1 \cdot \text{curl } E_2)$$

$$= (i \omega / c) (B_1 \cdot H_2 - H_1 \cdot B_2) + (i \omega / c) (E_1 \cdot D_2 - D_1 \cdot E_2)$$

$$+ (4 \pi / c) (\mathbf{j}_{x,1} \cdot E_2 - \mathbf{j}_{x,2} \cdot E_1).$$

But $B_1 \cdot H_2 = \mu_{ik} H_{ik} H_{2i} = H_1 \cdot B_2$, and $E_1 \cdot D_2 = D_1 \cdot E_2$, so that the first two terms on the right-hand side are zero. The left-hand side can be transformed by a formula of vector analysis, and the result is

$$\text{div } [E_1 \times H_2 - E_2 \times H_1] = (4 \pi / c) (\mathbf{j}_{x,1} \cdot E_2 - \mathbf{j}_{x,2} \cdot E_1).$$

We integrate this equation over all space; the integral on the left-hand side can be transformed into one over an infinitely remote surface, and is zero. Thus we have

$$\int \mathbf{j}_{x,1} \cdot E_2 dV_1 = \int \mathbf{j}_{x,2} \cdot E_1 dV_2.$$

(89.2)

The integrals are taken only over the volumes of sources 1 and 2 respectively, since the currents $\mathbf{j}_{x,1}$ and $\mathbf{j}_{x,2}$ are zero elsewhere. Since the wires are thin, the effect of each on the