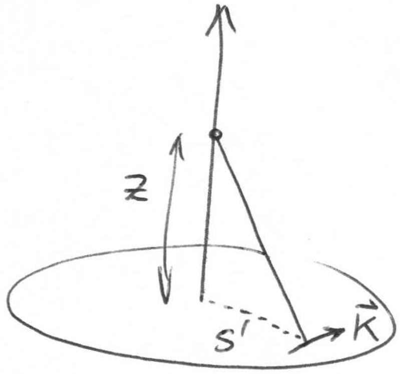
Homework solns HW #1 PH 632



$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, dA'$$

To evaluate cross prod, put $\vec{r}\,'$ on x-axis so that

$$\vec{r}-\vec{r}\,' = z\hat{z} - s'\hat{x}$$

and $\vec{K} = \sigma\omega s'\hat{y}$

Then

$$\vec{K}\times(\vec{r}-\vec{r}\,') = \sigma\omega s' z(\hat{y}\times\hat{z}) - \sigma\omega s'^2(\hat{y}\times\hat{x})$$

$$= \sigma\omega s' z\,\hat{x} + \sigma\omega s'^2\,\hat{z}$$

By symmetry I expect the final $\vec{B}$ field to point on z-axis (other directions would break the rotational symm about the z-axis)

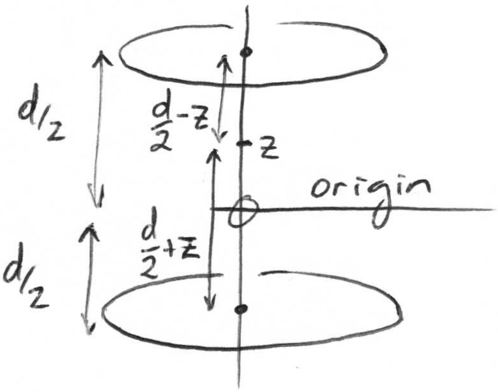
$$\vec{B} = \left(\frac{\mu_0}{4\pi}\int \frac{\sigma\omega s'^2}{|\vec{r}-\vec{r}\,'|^3}\, dA\right)\hat{z}$$

$$B_z = \frac{\mu_0\sigma\omega}{4\pi}\int_{s'=0}^{R} \frac{s'^2}{(z^2+s'^2)^{3/2}}\, 2\pi s'\, ds'$$

$$= \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{s'^2}{(z^2+s'^2)^{3/2}}\, ds'$$

$$= \frac{\mu_0\sigma\omega}{2}\left[\frac{s'^2+2z^2}{\sqrt{z^2+s'^2}}\right]_0^R = \frac{\mu_0\sigma\omega}{2}\left(\frac{R^2+2z^2}{\sqrt{z^2+R^2}} - \frac{2z^2}{\sqrt{z^2}}\right)$$

$$= \frac{\mu_0\sigma\omega}{2}\left(\frac{R^2+2z^2}{\sqrt{z^2+R^2}} - 2z\right)$$



The B-field at point z

$$B_z = \frac{\mu_0 I}{2}\left[\frac{R^2}{\left(R^2+\left(\frac{d}{2}-z\right)^2\right)^{3/2}} + \frac{R^2}{\left(R^2+\left(\frac{d}{2}+z\right)^2\right)^{3/2}}\right]$$

$$= \frac{\mu_0 I R^2}{2}\left[\left(R^2+\left(\frac{d}{2}-z\right)^2\right)^{-3/2} + \left(R^2+\left(\frac{d}{2}+z\right)^2\right)^{-3/2}\right]$$

$$\frac{\partial B_z}{\partial z} = \frac{\mu_0 I R^2}{2}\left[-\frac{3}{\cancel{2}}\left(R^2+\left(\frac{d}{2}-z\right)^2\right)^{-5/2}\cancel{2}\left(\frac{d}{2}-z\right)(-1) - \frac{3}{\cancel{2}}\left(R^2+\left(\frac{d}{2}+z\right)^2\right)^{-5/2}\cancel{2}\left(\frac{d}{2}+z\right)\right]$$

when $z = 0$

$$\frac{\partial B_z}{\partial z} = \frac{\mu_0 I R^2}{2}\left[+\frac{3}{\cancel{2}}\left(R^2+\frac{d^2}{4}\right)^{-5/2}\cancel{2}\left(\frac{d}{2}\right) - \frac{3}{\cancel{2}}\left(R^2+\frac{d^2}{4}\right)\cancel{2}\left(\frac{d}{2}\right)\right]$$

$$= 0.$$

Now compute the second derivative

$$\frac{\partial B_z}{\partial z} = \frac{3\mu_0 I R^2}{2}\left[+\left(R^2+\left(\frac{d}{2}-z\right)^2\right)^{-5/2}\left(\frac{d}{2}-z\right) - \left(R^2+\left(\frac{d}{2}+z\right)^2\right)^{-5/2}\left(\frac{d}{2}+z\right)\right]$$

$$\frac{\partial^2 B_z}{\partial z^2} = \frac{3\mu_0 I R^2}{2}\Bigg[-\left(R^2+\left(\frac{d}{2}-z\right)^2\right)^{-5/2} + \frac{5}{2}\left(R^2+\left(\frac{d}{2}-z\right)^2\right)^{-7/2} 2\left(\frac{d}{2}-z\right)^2$$

$$- \left(R^2+\left(\frac{d}{2}+z\right)^2\right)^{-5/2} + \frac{5}{2}\left(R^2+\left(\frac{d}{2}+z\right)^2\right)^{-7/2}\left(\frac{d}{2}+z\right)^2\Bigg]$$

Now setting $z=0$

$$\left.\frac{\partial^2 B_z}{\partial z^2}\right|_{z=0} = \frac{6\mu_0 I R^2}{2} \underbrace{\left[-\left(R^2 + \frac{d^2}{4}\right)^{-5/2} + 5\left(R^2 + \frac{d^2}{4}\right)^{-7/2} \frac{d^2}{4} \right]}_{\substack{=0 \\ \text{when } d=R.}}$$

When $d=R$ we have

$$\boxed{B_z(z=0) = \frac{8\mu_0 I}{5\sqrt{5}\,R}}$$