Chapter 6. Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

6.1 Displacement Current

The magnetic field due to a current distribution satisfies Ampere’s law,

\[ \nabla \times \mathbf{H} = \mathbf{J} \]  \hspace{1cm} (5.45)

Using Stokes’s theorem we can transform this equation into an integral form:

\[ \oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot \mathbf{n} \, da \]  \hspace{1cm} (5.47)

We shall examine this law, show that it sometimes fails, and find a generalization that is always valid.

Magnetic field near a charging capacitor

Consider the circuit shown in Fig. 6.1, which consists of a parallel plate capacitor being charged by a constant current \( I \). Ampere’s law applied to the loop \( C \) and the surface \( S_1 \) leads to

\[ \int_C \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot \mathbf{n} \, da = I \]  \hspace{1cm} (6.1)

If, on the other hand, Ampere’s law is applied to the loop \( C \) and the surface \( S_2 \), we find

\[ \int_C \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J} \cdot \mathbf{n} \, da = 0 \]  \hspace{1cm} (6.2)

Equations 6.1 and 6.2 contradict each other and thus cannot both be correct. The origin of this contradiction is that Ampere’s law is a faulty equation, not consistent with charge conservation:

\[ \int_{S_2} \mathbf{J} \cdot \mathbf{n} \, da - \int_{S_1} \mathbf{J} \cdot \mathbf{n} \, da = \int_{S_1+S_2} \mathbf{J} \cdot \mathbf{n} \, da = -\int \frac{\partial \rho}{\partial t} \, dv \neq 0 \]  \hspace{1cm} (6.3)
Generalization of Ampere’s law by Maxwell

Ampere’s law was derived for steady-state current phenomena with $\mathbf{V} \cdot \mathbf{J} = 0$. Using the continuity equation for charge and current

$$\mathbf{V} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (6.4)$$

and Coulomb’s law

$$\mathbf{V} \cdot \mathbf{D} = \rho \quad (6.5)$$

we obtain

$$\mathbf{V} \cdot \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \quad (6.6)$$

Ampere’s law is generalized by the replacement

$$\mathbf{J} \rightarrow \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (6.7)$$

i.e.,

$$\mathbf{V} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (6.8)$$

Maxwell equations

Maxwell equations consist of Eq. 6.8 plus three equations with which we are already familiar:

$$\mathbf{V} \cdot \mathbf{D} = \rho \quad \mathbf{V} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (6.9)$$

$$\mathbf{V} \cdot \mathbf{B} = 0 \quad \mathbf{V} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (6.10)$$

6.2 Vector and Scalar Potentials

Definition of $\mathbf{A}$ and $\Phi$

Since $\mathbf{V} \cdot \mathbf{B} = 0$, we can still define $\mathbf{B}$ in terms of a vector potential:

$$\mathbf{B} = \mathbf{V} \times \mathbf{A} \quad (6.10)$$

Then Faraday’s law can be written

$$\mathbf{V} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{V} \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (6.11)$$

The vanishing curl means that we can define a scalar potential $\Phi$ satisfying

$$-\nabla \Phi = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \quad (6.11)$$

or

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \quad (6.12)$$
Maxwell equations in terms of vector and scalar potentials

$\mathbf{E}$ and $\mathbf{B}$ defined as Eqs. 6.10 and 6.12 automatically satisfy the two homogeneous Maxwell equations. At this stage we restrict our considerations to the vacuum form of the Maxwell equations, i.e., $\mathbf{H} = \mathbf{B}/\mu_0$ and $\mathbf{D} = \varepsilon_0 \mathbf{E}$. Then the two inhomogeneous equations become

$$\nabla^2 \Phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0} \tag{6.13}$$

and

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J} \tag{6.14}$$

where $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum. The four first order coupled differential equations (Maxwell equations) reduce to two second order differential equations, but they are still coupled.

Lorenz condition and wave equations

We can uncouple the two inhomogeneous equations by exploiting the arbitrariness of the potentials. $\mathbf{B}$ is unchanged by the transformation,

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \Lambda \tag{6.15}$$

where $\Lambda$ is a scalar function. The scalar function must be simultaneously transformed,

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \tag{6.16}$$

for the electric field to be unchanged. Until now only $\nabla \times \mathbf{A}$ has been specified; the choice of $\nabla \cdot \mathbf{A}$ is still arbitrary. Imposing the so-called Lorenz condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \tag{6.17}$$

decouples Eqs. 6.13 and 6.14 and results in a considerable simplification:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{6.18}$$

and

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \tag{6.19}$$

Thus, the Lorentz condition makes $\mathbf{A}$ and $\Phi$ satisfy inhomogeneous wave equations of similar forms.
6.3 Gauge Transformations: Lorentz Gauge and Coulomb Gauge

Gauge transformation (Eqs. 6.15 and 6.16):
\[
\begin{align*}
A & \rightarrow A' = A + \nabla \Lambda \\
\Phi & \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}
\end{align*}
\]

Gauge invariance:
\[
\begin{align*}
B &= \nabla \times A = \nabla \times A' \\
E &= -\nabla \phi - \frac{\partial A}{\partial t} = -\nabla \phi' - \frac{\partial A'}{\partial t}
\end{align*}
\]

(6.20)

We can construct potentials \((A' \text{ and } \Phi')\) satisfying the Lorentz condition (Eq. 6.17) from potentials \((A \text{ and } \Phi)\) not satisfying the condition via gauge transformation,
\[
\nabla \cdot A' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = \nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0,
\]

(6.21)

where a gauge function \(\Lambda\) satisfies
\[
\nabla^2 \Lambda - \frac{\partial^2 \Lambda}{\partial t^2} = -\left(\nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t}\right)
\]

(6.22)

Lorentz gauge

All potentials satisfying the restricted gauge transformation
\[
\begin{align*}
A & \rightarrow A + \nabla \Lambda \\
\Phi & \rightarrow \Phi - \frac{\partial \Lambda}{\partial t}
\end{align*}
\]

(6.23)

where
\[
\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0
\]

(6.24)

are said to belong to the Lorentz gauge.

Coulomb gauge (radiation or transverse gauge)

The Coulomb gauge is defined by the gauge fixing condition
\[
\nabla \cdot A = 0
\]

(6.25)

In this gauge, the scalar potential satisfies the Poisson equation (see Eq. 6.13),
\[
\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}
\]

(6.26)

and the potential can be expressed in terms of instantaneous values of the charge density,
\[
\Phi(x, t) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(x', t)}{|x - x'|} d^3x'
\]

(6.27)
The vector potential satisfies the inhomogeneous wave equation,
\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\mu_0 j + \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} \]  
(6.28)

The current density can be split into two terms,
\[ J = J_l + J_t \]  
(6.29)

where
\[ J_l : \text{longitudinal or irrotational current} \quad \nabla \times J_l = 0 \]
\[ J_t : \text{transverse or solenoidal current} \quad \nabla \cdot J_t = 0 \]

The longitudinal current can be written as
\[ J_l(x, t) = \int J_l(x', t) \delta(x - x') \, d^3x' = -\frac{1}{4\pi} \int J_l(x', t) \nabla^2 \frac{1}{|x - x'|} \, d^3x' \]
\[ = -\frac{1}{4\pi} \int J_l(x', t) \nabla' \cdot \left( \frac{1}{|x - x'|} \right) \, d^3x' \]
\[ = -\frac{1}{4\pi} \int \left( J_l(x', t) \nabla' \frac{1}{|x - x'|} \right) \cdot n' \, da' - \frac{1}{4\pi} \int \frac{\nabla' \cdot J_l(x', t)}{|x - x'|} \, d^3x' \]

The surface integral vanishes for \( r' \to \infty \) and \( \nabla' \cdot J_l(x', t) = \nabla' \cdot J(x', t) \), therefore
\[ J_l(x, t) = -\frac{1}{4\pi} \int \frac{\nabla' \cdot J(x', t)}{|x - x'|} \, d^3x' \]  
(6.29)

Using the continuity equation, we can write
\[ \mu_0 J_l(x, t) = \frac{\mu_0}{4\pi} \int \frac{\partial \rho(x', t)}{\partial t} \, d^3x' = \frac{1}{c^2} \nabla \frac{\partial}{\partial t} \left[ \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(x', t)}{|x - x'|} \, d^3x' \right] = \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} \]

Thus, the wave equation (Eq. 6.28) becomes
\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\mu_0 J_l \]  
(6.30)

where
\[ J_l(x, t) = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{J(x', t)}{|x - x'|} \, d^3x' \]  
(6.31)

The Coulomb gauge is often used when no sources are present. Then, the fields are given by
\[ E = -\frac{\partial A}{\partial t} \]  
(6.32)
\[ B = \nabla \times A \]
6.4 Green Functions for the Wave Equation

The wave equations (Eqs. 18, 19, 30) have the basic structure

\[ \nabla^2 \Psi(x, t) - \frac{1}{c^2} \frac{\partial^2 \Psi(x, t)}{\partial t^2} = -4\pi f(x, t) \quad (6.33) \]

where \( f(x, t) \) is a source distribution. The factor \( c \) is the velocity of propagation in the medium, assumed to be without dispersion, i.e., \( c \) is independent of frequency. We remove the explicit time dependence of the wave equation introducing Fourier transformation.

**Fourier transformation and inverse Fourier transformation**

\[
\begin{align*}
\Psi(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(x, \omega) e^{-i\omega t} \, d\omega \\
\Psi(x, \omega) &= \int_{-\infty}^{\infty} \Psi(x, t) e^{i\omega t} \, dt \\
f(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x, \omega) e^{-i\omega t} \, d\omega \\
f(x, \omega) &= \int_{-\infty}^{\infty} f(x, t) e^{i\omega t} \, dt
\end{align*}
\]  

(6.34)

In the frequency domain, the wave equation (Eq. 6.33) becomes

\[ (\nabla^2 + k^2)\Psi(x, \omega) = -4\pi f(x, \omega) \quad (6.35) \]

where \( k = \omega/c \) is the wave number associate with frequency \( \omega \).

**Green function for \( k \)**

The Green function appropriate to Eq. 6.35 satisfies the inhomogeneous equation

\[ (\nabla^2 + k^2)G(x, x') = -4\pi \delta(x - x') \quad (6.36) \]

If there is no boundary surfaces, the Green function depend only on \( R = |R| = |x - x'| \). Then, Eq. 6.36 becomes

\[ \frac{1}{R} \frac{d}{dR} \left( RG_k \right) + k^2 G_k = -4\pi \delta(R) \quad (6.37) \]

For \( R \neq 0 \),

\[ \frac{d}{dR} \left( RG_k \right) + k^2 RG_k = 0 \]

has the solution

\[ RG_k(R) = Ae^{ikR} + Be^{-ikR} \]

In the limit of electrostatics, \( kR \ll 1 \),

\[ \lim_{kR \rightarrow 0} G_k(R) = \frac{1}{R} \quad (6.38) \]

thus,

\[ G_k(R) = AG_k^{(+)}(R) + BG_k^{(-)}(R) \quad (6.39) \]
where

\[ G_k^{(±)}(R) = \frac{e^{±ikR}}{R} \quad (6.40) \]

with \( A + B = 1 \). \( G_k^{(±)}(R) \) represent diverging and converging spherical waves, respectively.

**Time dependent Green function**

The time dependent Green functions corresponding to \( G_k^{(±)}(R) \) satisfy

\[ \left( \nabla_x^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^{(±)}(x, t; x', t') = -4\pi \delta(x - x') \delta(t - t') \quad (6.41) \]

Fourier transform of Eq. 6.41 leads to

\[ \int_{-∞}^{∞} \left( \nabla_x^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^{(±)}(x, t; x', t') e^{i\omega t} dt = -4\pi \delta(x - x') \int_{-∞}^{∞} \delta(t - t') e^{i\omega t'} dt \]

\[ \Rightarrow \nabla_x^2 G^{(±)}(x, \omega; x', t') + k^2 G^{(±)}(x, \omega; x', t') = -4\pi \delta(x - x') e^{i\omega t'} \quad (6.42) \]

The solutions of Eq. 6.42 are, therefore,

\[ G^{(±)}(x, \omega; x', t') = G_k^{(±)}(R) e^{i\omega t'} \quad (6.43) \]

Inverse Fourier transform of Eq. 6.43 gives rise to

\[ G^{(±)}(x, t; x', t') = G^{(±)}(R, \tau) = \frac{1}{2\pi} \int_{-∞}^{∞} \frac{e^{±ikR}}{R} e^{-i\omega \tau} d\omega \quad (6.44) \]

where \( \tau = t - t' \) is the relative time. Since \( k = \omega/c \),

\[ G^{(±)}(R, \tau) = \frac{1}{R} \delta \left( \tau + \frac{R}{c} \right) \quad (6.45) \]

or, more explicitly,

\[ G^{(±)}(x, t; x', t') = \frac{1}{|x - x'|} \delta \left( t' - t \mp \frac{|x - x'|}{c} \right) \quad (6.46) \]

**Retarded and advanced Green functions**

\[ G^{(+)}: \text{retarded Green function} \quad \text{An event observed at } t \text{ due to the source a distance } R \text{ away at a retarded time } t' = t - R/c. \]

\[ G^{(-)}: \text{advanced Green function} \quad \text{Represents an advanced time } t' = t + R/c \]

When a source distribution \( f(x', t') \) is localized in time and space, we can envision two limiting situations:

(i) At time \( t \to -∞ \), there exists a wave \( \Psi_{in}(x, t) \) satisfying the homogeneous equation.

\[ \Psi(x, t) = \Psi_{in}(x, t) + \int \int G^{(±)}(x, t; x', t') f(x', t') \, d^3x' \, dt' \quad (6.47) \]

\( G^{(±)} \) ensures no contribution before the source is activated.
(ii) At time \( t \to +\infty \), \( \Psi_{out}(x, t) \) exists.

\[
\Psi(x, t) = \Psi_{out}(x, t) + \iint G^{(-)}(x, t; x', t') f(x', t') \, d^3 x' \, dt' \tag{6.48}
\]

\( G^{(-)} \) makes sure no contribution after the source shuts off.

With the initial condition \( \Psi_{in}(x, t) = 0 \), we can write the general solution

\[
\Psi(x, t) = \int \frac{[f(x', t')]}{|x - x'|} \, d^3 x' \tag{6.49}
\]

where \([ ]_{\text{ret}}\) means \( t' = t - R/c \).

6.5 Retarded Solutions

Using Eq. 6.49 we obtain the solutions of the wave equations Eq. 6.18 and 6.19: the retarded scalar potential is expressed as

\[
\Phi(x, t) = \frac{1}{4 \pi \epsilon_0} \int \frac{[\rho(x', t')]}{R} \, d^3 x' \tag{6.50}
\]

and the retarded vector potential is written as

\[
A(x, t) = \frac{\mu_0}{4 \pi} \int \frac{[J(x', t')]}{R} \, d^3 x' \tag{6.51}
\]

where \( R = |R| = |x - x'| \) and \( t' = t - R/c \). Eqs. 6.50 and 6.51 indicate that at a given point \( x \) and a given time \( t \) the potentials are determined by the charge and current that existed at other points in space \( x' \) at earlier times \( t' \). The time appropriate to each source point is earlier than \( t \) by \( R/c \), the time required to travel from source to field point \( x \) with velocity \( c \). In the above procedure it is essential to impose the Lorentz condition on the potentials; otherwise it would not be the simple wave equations.

Retarded solutions for the fields

Using the wave equations for the potentials (Eq. 6.18 and 6.19) and the definitions of the fields (Eq. 6.10 and 6.12)

\[
E = -\nabla \Phi - \frac{\partial A}{\partial t}, \quad B = \nabla \times A
\]

we obtain

\[
\nabla^2 E(x, t) - \frac{1}{c^2} \frac{\partial^2 E(x, t)}{\partial t^2} = -\frac{1}{\epsilon_0} \left( -\nabla \rho - \frac{1}{c^2} \frac{\partial J}{\partial t} \right) \tag{6.52}
\]

and

\[
\nabla^2 B(x, t) - \frac{1}{c^2} \frac{\partial^2 B(x, t)}{\partial t^2} = -\mu_0 \nabla \times J \tag{6.53}
\]
Then the retarded solutions for the fields can immediately be written as

\[
E(x, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{R} \left[ -\nabla'\rho - \frac{1}{c^2} \frac{\partial J}{\partial t'} \right]_{\text{ret}} d^3x' \tag{6.54}
\]

and

\[
B(x, t) = \frac{\mu_0}{4\pi} \int \frac{1}{R} \left[ \nabla' \times J \right]_{\text{ret}} d^3x' \tag{6.55}
\]

where

\[
[\nabla' \rho]_{\text{ret}} = \nabla' [\rho]_{\text{ret}} - \left[ \frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \nabla' \left( t - \frac{R}{c} \right) = \nabla' [\rho]_{\text{ret}} - \frac{\hat{R}}{c} \left[ \frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \tag{6.56}
\]

and

\[
[\nabla' \times J]_{\text{ret}} = \nabla' \times [J]_{\text{ret}} + \left[ \frac{\partial J}{\partial t'} \right]_{\text{ret}} \nabla' \left( t - \frac{R}{c} \right) = \nabla' \times [J]_{\text{ret}} + \left[ \frac{\partial J}{\partial t'} \right]_{\text{ret}} \hat{R} \tag{6.57}
\]

with \( \hat{R} = R/R \). Inserting Eqs. 6.56 and 6.57 into Eqs. 6.54 and 6.55 and applying integration by parts,

\[
\int \frac{1}{R} \nabla' [\rho]_{\text{ret}} d^3x' = \int \nabla' \left( \frac{[\rho]_{\text{ret}}}{R} \right) d^3x' - \int \nabla' \left( \frac{1}{R} \right) [\rho]_{\text{ret}} d^3x' = -\int \frac{\hat{R}}{R^2} [\rho]_{\text{ret}} d^3x'
\]

and

\[
\int \frac{1}{R} \nabla' \times [J]_{\text{ret}} d^3x' = \int \nabla' \times \left( \frac{[J]_{\text{ret}}}{R} \right) d^3x' - \int \nabla' \left( \frac{1}{R} \right) \times [J]_{\text{ret}} d^3x' = \int [J]_{\text{ret}} \times \frac{\hat{R}}{R^2} d^3x'
\]

we get

\[
E(x, t) = \frac{1}{4\pi\varepsilon_0} \int \left\{ \frac{\hat{R}}{R^2} [\rho(x', t')]_{\text{ret}} + \frac{\hat{R}}{cR} \left[ \frac{\partial \rho(x', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2R} \left[ \frac{\partial [J(x', t')]}{\partial t'} \right]_{\text{ret}} \right\} d^3x' \tag{6.58}
\]

and

\[
B(x, t) = \frac{\mu_0}{4\pi} \int \left\{ \frac{\hat{R}}{R^2} [J(x', t')]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\} d^3x' \tag{6.59}
\]

If the charge and current densities are independent of time, Eqs. 6.58 and 6.59 reduce to the Coulomb and Biot-Savart laws.

### 6.6 Maxwell Equations in Matter

**Microscopic Maxwell equations**

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \quad \nabla \cdot \mathbf{D} = \rho \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0
\]

**Macroscopic Maxwell equations**

The macroscopic Maxwell equations are obtained by taking spatial average of the microscopic Maxwell equations.
The spatial average of a function $F(x, t)$ with respect to a test function $f(x)$ is defined as
\[
\langle F(x, t) \rangle = \int f(x') F(x - x', t) \, d^3 x'
\] (6.61)

Accordingly,
\[
E(x, t) = \langle e(x, t) \rangle \\
B(x, t) = \langle b(x, t) \rangle
\] (6.62)

and the two homogeneous equations become
\[
\langle \nabla \cdot b \rangle = 0 \Rightarrow \nabla \cdot B = 0 \tag{6.63}
\]
\[
\langle \nabla \times e + \partial b / \partial t \rangle = 0 \Rightarrow \nabla \times E + \partial B / \partial t = 0
\]

The averaged inhomogeneous equations become
\[
\varepsilon_0 \nabla \cdot E = \langle \eta(x, t) \rangle \tag{6.64}
\]
\[
\frac{1}{\mu_0} \nabla \times B - \varepsilon_0 \frac{\partial E}{\partial t} = \langle j(x, t) \rangle
\]

We can decompose the microscopic charge density $\eta$ as
\[
\eta = \eta_{\text{free}} + \eta_{\text{bound}}
\] (6.65)

The averaged microscopic charge density reduces to
\[
\langle \eta(x, t) \rangle = \rho(x, t) + \rho_b(x, t) + \cdots = \rho(x, t) - \nabla \cdot P(x, t) + \cdots \tag{6.66}
\]

where $\rho$ is the macroscopic charge density,
\[
\rho(x, t) = \left( \sum_{j(\text{free})} q_j \delta(x - x_j) + \sum_{n(\text{molecule})} q_n \delta(x - x_n) \right) \tag{6.67}
\]

and $P$ is the macroscopic polarization,
\[
P(x, t) = \left( \sum_{n(\text{molecule})} p_n \delta(x - x_n) \right) \tag{6.68}
\]
Similarly, the averaged microscopic current density can be expressed as

\[
\langle j(x, t) \rangle = J + J_b + J_p + \cdots = J(x, t) + \nabla \times M(x, t) + \frac{\partial P(x, t)}{\partial t} + \cdots
\] (6.69)

where \(J\) is the macroscopic current density,

\[
J(x, t) = \sum_{j(\text{tree})} q_j v_j \delta(x - x_j) + \sum_{n(\text{molecule})} q_n v_n \delta(x - x_n)
\] (6.70)

and \(M\) is the macroscopic magnetization,

\[
M(x, t) = \sum_{n(\text{molecule})} m_n \delta(x - x_n)
\] (6.71)

The polarization current density

\[
J_p = \frac{\partial P(x, t)}{\partial t}
\] (6.72)

involves a flow of bound charge as shown in Fig. 6.3. If \(P\) increases, the charge on each end increases, giving a net current

\[
dl = \frac{\partial Q}{\partial t} = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp
\]

\[ \begin{array}{c}
\text{Fig 6.3} \\
\end{array} \]

Inserting Eqs. 6.66 and 6.69 into Eq. 6.64, we obtain the macroscopic inhomogeneous equations:

\[
\varepsilon_0 \nabla \cdot E = \rho - \nabla \cdot P \quad \rightarrow \quad \nabla \cdot (\varepsilon_0 E + P) = \rho \quad \rightarrow \quad \nabla \cdot D = \rho
\]

where \(D \equiv \varepsilon_0 E + P\) and

\[
\frac{1}{\mu_0} \nabla \times B - \varepsilon_0 \frac{\partial E}{\partial t} = J + \nabla \times M + \frac{\partial P}{\partial t} \quad \rightarrow \quad \nabla \times \left( \frac{B}{\mu_0} - M \right) = J + \frac{\partial}{\partial t} (\varepsilon_0 E + P)
\]

\[
\rightarrow \nabla \times H = J + \frac{\partial D}{\partial t}
\]

where \(H \equiv B/\mu_0 - M\).
6.7 Poynting’s Theorem

Energy conservation: Poynting vector

It is simple to derive the energy conservation laws from Maxwell equations. We take the scalar product of one equation with $\mathbf{E}$ and of another with $\mathbf{H}$:

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$  \hspace{1cm} (6.73)

and

$$\mathbf{H} \cdot \nabla \times \mathbf{E} = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$$  \hspace{1cm} (6.74)

Subtracting Eq. 6.74 from Eq. 7.73 and using the vector identity,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}$$

we have

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\mathbf{J} \cdot \mathbf{E}$$  \hspace{1cm} (6.75)

We assume that (i) the macroscopic medium is linear and dispersionless, (i.e., $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ with $\varepsilon$ and $\mu$ real and frequency independent) and (ii) the total electromagnetic energy density is denoted by

$$u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$$

and define the vector $\mathbf{S}$, called the Poynting vector, by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$  \hspace{1cm} (6.77)

Then we can rewrite Eq. 6.75 as

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} = -\mathbf{J} \cdot \mathbf{E}$$  \hspace{1cm} (6.78)

Integrating over any volume, we have

$$\int_S \mathbf{S} \cdot \mathbf{n} \, da = -\frac{d}{dt} \int_V u \, dv - \int_V \mathbf{J} \cdot \mathbf{E} \, dv$$  \hspace{1cm} (6.79)

Physical meaning of the three terms in Eq. 6.79

(i) For a single charge $q$ the rate of doing work by external electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ is $q \mathbf{v} \cdot \mathbf{E}$, where $\mathbf{v}$ is the velocity of the charge (no work is done by $\mathbf{B}$ because $\mathbf{F}_m \perp \mathbf{v}$). For a continuous charge and current distribution, the total rate of doing work by the fields in a finite volume $V$ is

$$\int_V \mathbf{J} \cdot \mathbf{E} \, dv$$

In other words, it is the rate of increase of mechanical energy of charges.

(ii) The second term

$$\frac{d}{dt} \int_V u \, dv$$

Is the rate at which field energy within the volume is increasing.
Finally, the first term has the appearance of an outgoing flux and must be equal to the rate at which energy is leaving the volume per unit time. Therefore, it is evident that $S$ is a vector pointing along the direction in which energy is flowing and whose magnitude is equal to the flux of energy per unit time through a unit area normal to itself.

**Momentum conservation: momentum density and Maxwell stress tensor**

The electromagnetic force acting on a charged particle is

$$ F = q(E + v \times B) $$

Denoting $P_{\text{mech}}$ as the total momentum of all the particles in the volume $V$, we can write

$$ \frac{dP_{\text{mech}}}{dt} = \int_V (\rho E + J \times B) \, dv $$

(6.80)

Using Maxwell equations, we modify the integrand as

$$ \rho E + J \times B = \varepsilon_0 \left[ E(\nabla \cdot E) + B \frac{\partial E}{\partial t} - c^2 B \times (\nabla \times B) \right] $$

$$ = \varepsilon_0 \left[ E(\nabla \cdot E) + c^2 B(\nabla \cdot B) - E \times (\nabla \times E) - c^2 B \times (\nabla \times B) \right] - \varepsilon_0 \frac{\partial}{\partial t} (E \times B) $$

Then Eq. 6.80 becomes

$$ \frac{dP_{\text{mech}}}{dt} + \frac{d}{dt} \int_V \frac{1}{c^2} (E \times H) \, dv $$

$$ = \varepsilon_0 \int_V \left[ E(\nabla \cdot E) + c^2 B(\nabla \cdot B) - E \times (\nabla \times E) - c^2 B \times (\nabla \times B) \right] \, dv $$

(6.81)

We define a vector

$$ g = \frac{1}{c^2} (E \times H) = \frac{1}{c^2} S $$

(6.82)

(we will show that $g$ is the momentum density of the electromagnetic field) and introduce the three dimensional Maxwell stress tensor

$$ T_{\alpha\beta} = \varepsilon_0 \left[ E_{\alpha} E_{\beta} + c^2 p_{\alpha} p_{\beta} - \frac{1}{2} (E^2 + c^2 B^2) \delta_{\alpha\beta} \right], \quad \alpha, \beta = 1, 2, 3 $$

(6.83)

Then Eq. 6.81 becomes

$$ \frac{dP_{\text{mech}}}{dt} + \frac{d}{dt} \int_V g \, dv = \int_S T \cdot n \, da $$

(6.84)

where $n$ is the outward normal to the closed surface $S$. The tensor-vector product $T \cdot n$ is a vector and its $\alpha$ component is

$$ [T \cdot n]_{\alpha} = \sum_{\beta=1}^{3} T_{\alpha\beta} n_{\beta} $$

(6.85)

Physically, $T$ is the force per unit area (i.e., stress) acting on the surface. More precisely, $T_{\alpha\beta}$ is the force per unit area in the $\alpha$th direction acting on an element of surface oriented in the $\beta$th
direction; diagonal elements \((T_{11}, T_{22}, T_{33})\) represent pressures, and off-diagonal elements \((T_{12}, T_{23}, \text{etc.})\) are shears. Eq. 6.84 presents the momentum conservation of the electromagnetic system. The surface integral of Eq. 6.84 can be thought of as the total momentum flowing into our volume through the surface per unit time. Alternatively one might think of it as being the electromagnetic “force” exerted on our volume by the outside world. The second term on the left should be equal to the rate of change of the field momentum within the volume. This would indicate that \(g\) is the momentum density of the electromagnetic field.

**Poynting’s theorem for harmonic fields**

We assume that all fields and sources are monochromatic having a time dependence \(e^{-i\omega t}\)

\[
F(x, t) = \text{Re}[F(x)e^{-i\omega t}] = \frac{1}{2} [F(x)e^{-i\omega t} + F^*(x)e^{i\omega t}] 
\]

(6.86)

For product forms, we have

\[
F(x, t) \cdot G(x, t) = \frac{1}{4} [F^*(x) \cdot G(x) + F(x) \cdot G^*(x) + F(x) \cdot G(x)e^{-2i\omega t} + F^*(x) \cdot G^*(x)e^{2i\omega t}] 
\]

\[
= \frac{1}{2} \text{Re}[F^*(x) \cdot G(x) + F(x) \cdot G(x)e^{-2i\omega t}] 
\]

Then time average of the product is

\[
\langle F(x, t) \cdot G(x, t) \rangle = \text{Re} \left[ \frac{1}{2} F^*(x) \cdot G(x) \right] 
\]

(6.87)

**Maxwell equations for harmonic fields**

\[
\nabla \cdot D = \rho \quad \nabla \times H + i\omega D = J 
\]

(6.88)

\[
\nabla \cdot B = 0 \quad \nabla \times E - i\omega B = 0 
\]

where all the quantities are complex functions of \(x\).

**Work done by the fields**

We take the scalar product of one Maxwell equation with \(E\) and of another with \(B\):

\[
E \cdot \nabla \times H^* = J^* \cdot E + i\omega E \cdot D^* 
\]

(6.89)

and

\[
H^* \cdot \nabla \times E = i\omega B \cdot H^* 
\]

(6.90)

Subtracting Eq. 6.90 from Eq. 7.89 and using the vector identity,

\[
\nabla \cdot (E \times H^*) = H^* \cdot \nabla \times E - E \cdot \nabla \times H^* 
\]

(6.91)

we have

\[
\nabla \cdot (E \times H^*) = -i\omega(E \cdot D^* - B \cdot H^*) - J^* \cdot E 
\]

(6.92)

We now define the complex Poynting vector

\[
S = \frac{1}{2}(E \times H^*) 
\]

(6.93)
And the harmonic electric magnetic energy densities
\[ u_e = \frac{1}{4} (E \cdot D^*), \quad u_m = \frac{1}{4} (B \cdot H^*) \] (6.94)

Then the time-averaged rate of work done by the fields in the volume \( V \) have the relation
\[ \frac{1}{2} \int_V \mathbf{J}^* \cdot \mathbf{E} \, dv = -2i\omega \int_V (u_e - u_m) \, dv - \int_S \mathbf{S} \cdot \mathbf{n} \, da \] (6.95)

The real part of this equation gives the conservation of energy. If the energy densities are real (e.g., systems with lossless dielectrics and perfect conductors),
\[ \frac{1}{2} \int_V \text{Re}(\mathbf{J}^* \cdot \mathbf{E}) \, dv = -\int_S \text{Re}(\mathbf{S} \cdot \mathbf{n}) \, da \] (6.96)

which indicates that the steady-state, time-averaged rate of doing work on the sources in \( V \) by the fields is equal to the average flow of power into the volume \( V \) through the boundary surface \( S \).

### 6.8 Transformation Properties of Electromagnetic Fields

Maxwell equations are the fundamental law of physics and hence invariant under coordinate transformation such as rotation, inversion, and time reversal. The related physical quantities transform in a consistent fashion so that the form of the equations is the same as before.

**Inversion and time reversal**

**Polar vector and pseudovector**

Under the coordinate transformation of the pure inversion \((x \rightarrow x' = -x)\) polar vectors change sign and psuedovectors do not, while both behave exactly same under rotation, e.g.,

<table>
<thead>
<tr>
<th>(x)</th>
<th>polar vector</th>
<th>(v = \frac{dx}{dt})</th>
<th>polar vector</th>
<th>(L = x \times p)</th>
<th>pseudovector</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F = \frac{dp}{dt} = m \frac{d^2x}{dt^2})</td>
<td>polar vector</td>
<td>(N = x \times F)</td>
<td>pseudovector</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Time reversal**

A vector is either even or odd under the time reversal transformation \(t \rightarrow t' = -t\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>even</th>
<th>(v = \frac{dx}{dt})</th>
<th>odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F = \frac{dp}{dt} = m \frac{d^2x}{dt^2})</td>
<td>even</td>
<td>(p = mv)</td>
<td>odd</td>
</tr>
</tbody>
</table>
Electromagnetic quantities

Charge density $\rho$: It is experimentally proven that charge is a true scalar under all three coordinate transformations. Thus charge density $\rho$ is also a true scalar.

Current density $\mathbf{J}$: $\mathbf{J} = \rho \mathbf{v} \rightarrow \mathbf{J}$ is a polar vector and odd under time reversal.

Electric field $\mathbf{E}$: $\mathbf{F} = q \mathbf{E} \rightarrow \mathbf{E}$ is a polar vector and even under time reversal.

Magnetic field $\mathbf{B}$: $\mathbf{F} = q \mathbf{v} \times \mathbf{B} \rightarrow \mathbf{B}$ is a pseudovector and odd under time reversal.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$x \rightarrow x' = -x$</th>
<th>$t \rightarrow t' = -t$</th>
<th>$\mathbf{J}$</th>
<th>$x \rightarrow x' = -x$</th>
<th>$t \rightarrow t' = -t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>odd</td>
<td>$\mathbf{B}, \mathbf{M}, \mathbf{H}$</td>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>$\mathbf{E}, \mathbf{P}, \mathbf{D}$</td>
<td>odd</td>
<td>even</td>
<td>$\mathbf{B}, \mathbf{M}, \mathbf{H}$</td>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>$\mathbf{S} = \mathbf{E} \times \mathbf{H}$</td>
<td>odd</td>
<td>odd</td>
<td>$\mathbf{B}, \mathbf{M}, \mathbf{H}$</td>
<td>even</td>
<td>odd</td>
</tr>
</tbody>
</table>

From these facts, we see that the Maxwell equations are invariant under inversion and time reversal. For example, under inversion and time reversal

$$
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \rightarrow (-\nabla') \times \mathbf{H}' = (-\mathbf{J}') + \frac{\partial (-\mathbf{D}')}{\partial t} \rightarrow \nabla' \times \mathbf{H}' = \mathbf{J}' + \frac{\partial \mathbf{D}'}{\partial t}
$$

$$
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \rightarrow \nabla \times (-\mathbf{H}') = (-\mathbf{J}') + \frac{\partial \mathbf{D}'}{\partial t'} \rightarrow \nabla' \times \mathbf{H}' = \mathbf{J}' + \frac{\partial \mathbf{D}'}{\partial t}
$$

6.9 Magnetic Monopoles

A magnetic monopole is a hypothetical particle with only one magnetic pole (north or south). High energy physics theories such as string theory predict their existence. At present, however, there is no experimental evidence for the existence of magnetic monopoles or magnetic charges, which is explicitly stated in one of the Maxwell equations $\nabla \cdot \mathbf{B} = 0$. In fact, the Maxwell equations are asymmetric due to the “missing” terms of magnetic charge. Suppose that there exist magnetic charge and current densities, $\rho_m$ and $\mathbf{J}_m$, in addition to the electric densities, $\rho_e$ and $\mathbf{J}_e$, then the Maxwell equations in the Gaussian system (in which $\mathbf{E}$ and $\mathbf{B}$ have same unit) become

$$
\nabla \cdot \mathbf{E} = 4\pi \rho_e \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}_e + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad \nabla \cdot \mathbf{E} = 4\pi \rho_e \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}_e + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
$$

$$
\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \nabla \cdot \mathbf{B} = 4\pi \rho_m \quad \nabla \times \mathbf{B} = -\frac{4\pi}{c} \mathbf{J}_m - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}
$$

Magnetic charge would be conserved:

$$
\nabla \cdot \mathbf{J}_m + \frac{\partial \rho_m}{\partial t} = 0
$$

We deduce the transformation properties of $\rho_m$ and $\mathbf{J}_m$ from the known behavior of $\mathbf{E}$ and $\mathbf{B}$.

<table>
<thead>
<tr>
<th>$\rho_m$ pseudoscalar</th>
<th>Inversion</th>
<th>Time reversal</th>
<th>$\mathbf{J}_m$ pseudovector</th>
<th>Inversion</th>
<th>Time reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>odd</td>
<td>even</td>
<td>even</td>
<td>even</td>
<td>even</td>
</tr>
</tbody>
</table>
The opposite symmetries of $\rho_m$ and $\rho_e$ indicates that space inversion and time reversal may not be valid symmetries of the laws of physics in the realm of elementary particle physics.

**Dirac quantization condition**

Dirac showed that the existence of magnetic charge would explain why electric charge is quantized. Consider a system consisting of a single stationary magnetic monopole and a single moving electric monopole. The angular momentum of the fields

$$L_m = \frac{1}{c^2} \int x \times (E \times H) \, dv = \frac{q_e q_m}{4\pi} \frac{R}{R}$$

($R$: distance vector between the particles) is proportional to the product $q_e q_m$, and independent of the distance between them. Quantum mechanics dictates that angular momentum is quantized in units of $\hbar$ so that the product $q_e q_m$ must also be quantized. This means that if even a single magnetic monopole existed in the universe, and the form of Maxwell's equations is valid, all electric charges would then be quantized.