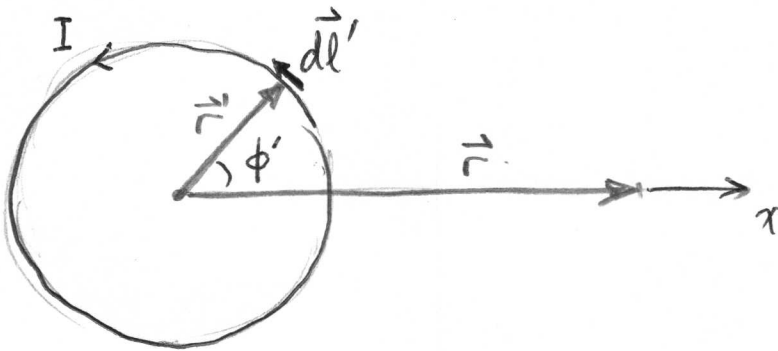Side view

top down view

The loop of current lies in the x-y plane.

Convenient to put field point, $\vec{r}$, in the x-z plane where $\phi = 0$.
(other planes that pass through z-axis will give
equivalent results)

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_{circle} \frac{\vec{dl'}}{|\vec{r} - \vec{r}'|}$$
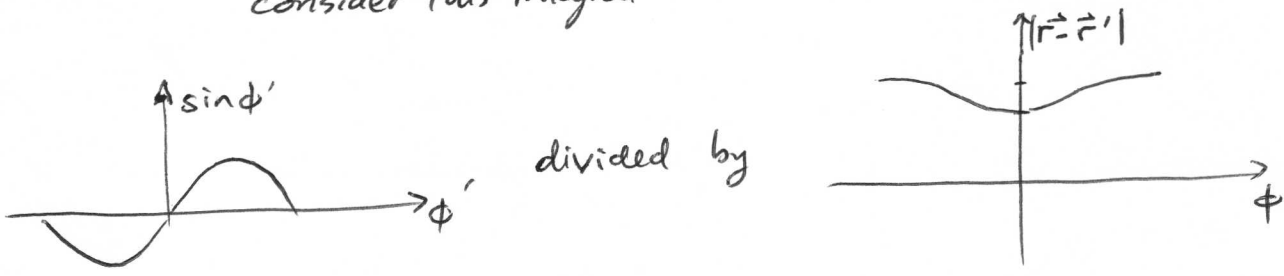
We are going to integrate a vector quantity:
- Break $\vec{dl'}$ into components
- Use Cartesian basis to describe the components.

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_{circle} \frac{R\,d\phi'\cos\phi'\,\hat{y} - R\,d\phi'\sin\phi'\,\hat{x}}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{R\mu_0 I}{4\pi} \int \frac{\sin\phi'}{|\vec{r}-\vec{r}'|}\,d\phi'\,\hat{x} + \frac{R\mu_0 I}{4\pi} \int \frac{\cos\phi'}{|\vec{r}-\vec{r}'|}\,d\phi'\,\hat{y}$$

consider this integral

divided by

odd fn divided by even fn.

$$\int (\text{odd fn})\,d\phi' = 0$$

$$\vec{A} = \frac{R\mu_0 I}{4\pi} \int \frac{\cos\phi'}{|\vec{r}-\vec{r}'|}\,d\phi'\,\hat{y}$$

Now we need expression for $|\vec{r}-\vec{r}'|$ in terms of ϕ' and the field ~~coordinat~~ point coordinates.

$$|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$= \sqrt{(r\sin\theta - R\cos\phi')^2 + (R\sin\phi')^2 + (r\cos\theta)^2}$$

$$= \sqrt{r^2 + R^2 - 2rR\cos\phi'\sin\theta}$$

We simplify this expression for $r \gg R$

$$|\vec{r} - \vec{r}'| \approx r \sqrt{1 - \frac{2R}{r} \cos\phi' \sin\theta}$$

This gives

$$\vec{A} = \frac{R\mu_0 I}{4\pi r} \int_0^{2\pi} \frac{\cos\phi'}{\sqrt{1 - \frac{2R}{r}\cos\phi'\sin\theta}} \, d\phi'$$

Taylor expansion of denominator gives

$$\vec{A} = \frac{R\mu_0 I}{4\pi r} \int_0^{2\pi} \cos\phi' \left(1 + \frac{R}{r}\cos\phi'\sin\theta\right) d\phi' \, \hat{y}$$

$$= \frac{R\mu_0 I}{4\pi r} \int_0^{2\pi} \frac{R}{r}\cos^2\phi'\sin\theta \, d\phi' \, \hat{y}$$

$$= \frac{\mu_0 I R^2}{4} \frac{1}{r^2} \sin\theta \, \hat{y}$$

This $\hat{y}$ direction is true when $\vec{r}$ is in x-z plane $(\phi = 0)$

More generally (when $\phi \neq 0$)

$$\boxed{\vec{A} = \frac{\mu_0 I R^2}{4} \frac{1}{r^2} \sin\theta \, \hat{\phi}}$$

Pop Quiz – Day 5

When a current loop with radius R is centered at the origin and lies in the x-y plane, the far-field vector potential is given by

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu_0 I R^2}{4} \frac{1}{r^2} \sin\theta \, \hat{\boldsymbol{\phi}}$$

a) Find $\mathbf{B}(\mathbf{r})$ in this far-field limit.

b) Sketch $\mathbf{B}(\mathbf{r})$ in the x-z plane.

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}};$ $\quad d\tau = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi}$

Curl: $\quad \nabla \times \mathbf{v} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \dfrac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$