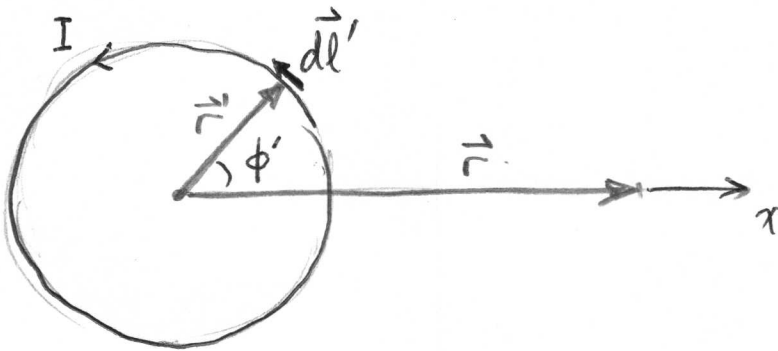


Side view



top down view

The loop of current lies in the x - y plane.

Convenient to put field point, \vec{r} , in the x - z plane where $\phi=0$.
(other planes that pass through z -axis will give equivalent results)

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_{\text{circle}} \frac{d\vec{l}'}{|\vec{r} - \vec{r}'|}$$

We are going to integrate a vector quantity:

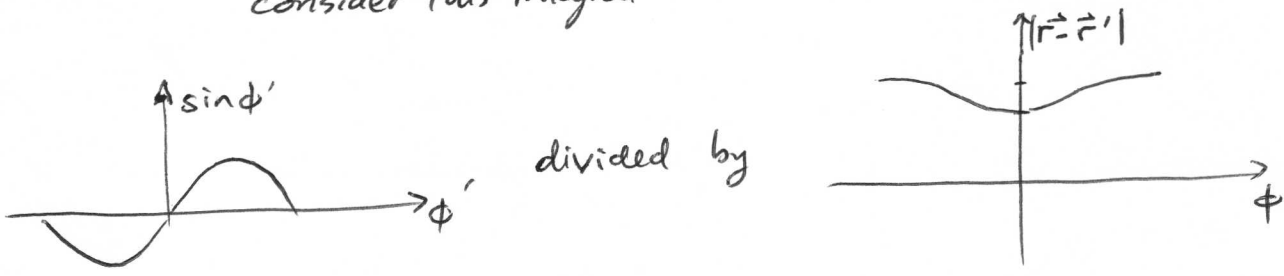
- Break $d\vec{l}'$ into components
- Use Cartesian basis to describe the components.

(2)

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_{\text{circle}} \frac{R d\phi' \cos \phi' \hat{y} - R d\phi' \sin \phi' \hat{x}}{|\vec{r} - \vec{r}'|}$$

$$= \underbrace{-\frac{R\mu_0 I}{4\pi} \int \frac{\sin \phi' d\phi'}{|\vec{r} - \vec{r}'|} \hat{x}}_{\text{consider this integral}} + \frac{R\mu_0 I}{4\pi} \int \frac{\cos \phi' d\phi'}{|\vec{r} - \vec{r}'|} \hat{y}$$

consider this integral



divided by

odd fn divided by even fn.

$$\int (\text{odd fn}) d\phi' = 0$$

$$\vec{A} = \frac{R\mu_0 I}{4\pi} \int \frac{\cos \phi' d\phi'}{|\vec{r} - \vec{r}'|} \hat{y}$$

Now we need expression for $|\vec{r} - \vec{r}'|$ in terms of ϕ' and the field ~~coordinates~~ point coordinates.

$$\begin{aligned} |\vec{r} - \vec{r}'| &= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\ &= \sqrt{(r \sin \theta - R \cos \phi')^2 + (R \sin \phi')^2 + (r \cos \theta)^2} \\ &= \sqrt{r^2 + R^2 - 2rR \cos \phi' \sin \theta} \end{aligned}$$

(3)

We simplify this expression for $r \gg R$

$$|\vec{r} - \vec{r}'| \approx r \sqrt{1 - \frac{2R}{r} \cos \phi' \sin \theta}$$

This gives

$$\vec{A} = \frac{R \mu_0 I}{4\pi r} \int_0^{2\pi} \frac{\cos \phi'}{\sqrt{1 - \frac{2R}{r} \cos \phi' \sin \theta}} d\phi'$$

Taylor expansion of denominator gives

$$\vec{A} = \frac{R \mu_0 I}{4\pi r} \int_0^{2\pi} \cos \phi' \left(1 + \frac{R}{r} \cos \phi' \sin \theta \right) d\phi' \quad \hat{y}$$

$$= \frac{R \mu_0 I}{4\pi r} \int_0^{2\pi} \frac{R}{r} \cos^2 \phi' \sin \theta d\phi' \quad \hat{y}$$

$$= \frac{\mu_0 I R^2}{4} \frac{1}{r^2} \sin \theta \quad \hat{y}$$

This \hat{y} direction is true when \vec{r} is in $x-z$ plane ($\phi=0$)

More generally (when $\phi \neq 0$)

$$\vec{A} = \frac{\mu_0 I R^2}{4} \frac{1}{r^2} \sin \theta \hat{\phi}$$

Pop Quiz - Day 5

When a current loop with radius R is centered at the origin and lies in the x-y plane, the far-field vector potential is given by

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu_0 I R^2}{4} \frac{1}{r^2} \sin \theta \hat{\phi}$$

- Find $\mathbf{B}(\mathbf{r})$ in this far-field limit.
- Sketch $\mathbf{B}(\mathbf{r})$ in the x-z plane.

Spherical. $d\mathbf{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$; $d\tau = r^2 \sin \theta dr d\theta d\phi$

Gradient: $\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r}$