PH 632

DAY 15

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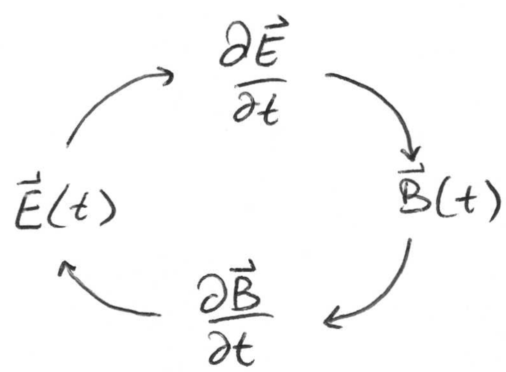
ELECTROMAGNETIC PLANE WAVES



Certain $\vec{E}$ & $\vec{B}$ fields can keep each other alive without the ~~help~~ help of any $\vec{J}$ or ρ.

In a vacuum we have

$$\nabla\cdot\vec{E} = 0 \quad , \quad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad , \quad \nabla\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Consider this pair of coupled differential eqns.

To solve this system of eqns, I need to substitute a constraint from the ~~first~~ $\nabla\times\vec{E}$ eq^n into the ~~second~~ $\nabla\times\vec{B}$ eq^n and vice versa.

$$\nabla\times\nabla\times\vec{E} = -\frac{\partial}{\partial t}\nabla\times B \quad , \quad \nabla\times\nabla\times\vec{B} = \mu_0\epsilon_0\frac{\partial}{\partial t}\nabla\times\vec{E}$$

There is a useful vector identity

$$\nabla \times \nabla \times \vec{V} = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$$

$$\Rightarrow \boxed{\begin{aligned} \nabla^2 \vec{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{—— ①} \\ \nabla^2 \vec{B} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t} \qquad \text{—— ②} \end{aligned}}$$

Equation ① & ② are eqns that relate 3-component vectors. Let's explode eqn ① into components

$$\left.\begin{aligned} \nabla^2 E_x &= \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \\ \nabla^2 E_y &= \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} \\ \nabla^2 E_z &= \mu_0 \epsilon_0 \frac{\partial^2 E_z}{\partial z^2} \end{aligned}\right\} \text{—— ①}$$

Here is an example of an electric field that satisfies ①

$$\vec{E}(x,y,z,t) = \begin{bmatrix} E_0 \sin(kz - \omega t) \\ 0 \\ 0 \end{bmatrix}$$

→ Pop Quiz

→ Visualizations.

Pop Quiz – Day 15

Maxwell's equations for **E** and **B** in a vacuum are satisfied by huge variety of **E** and **B** vector fields. One possible vector field for **E** is

$$E_x = E_0 \sin(kz - \omega t)$$

$$E_y = 0$$

$$E_z = 0$$

a) Maxwell's equations put a constraint on the ratio of ω/k. Find the ratio.

b) Find the unique **B**-field, $\mathbf{B}(x, y, z, t)$, that is compatible with the **E**-field (given above).

Hint: Your answer to part b will have $B_z = 0$.

Let's make the result more formal & general:

What is the general solution to 1 that has E_y & $E_z = 0$ and E_x only depends on z & t?

Answer: Use separation of variables

$$E_x(z,t) = Z(z)\,T(t)$$

$$\frac{\partial^2}{\partial z^2} T Z = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t} T Z$$

$$T \frac{\partial^2}{\partial z^2} Z = \mu_0 \epsilon_0 Z \frac{\partial^2}{\partial t^2} T$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = \mu_0 \epsilon_0 \frac{1}{T}\frac{d^2 T}{dt^2}$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2 \qquad\qquad \mu_0 \epsilon_0 \frac{1}{T}\frac{d^2 T}{dt^2} = -k^2$$

convenient name for the separation constant.

$$\frac{d^2 Z}{dz^2} = -k^2 Z \qquad\qquad \frac{d^2 T}{dt^2} = \frac{-k^2}{\mu_0 \epsilon_0} T$$