Chapter 3. Boundary-Value Problems in Electrostatics: Spherical and Cylindrical Geometries

3.1 Laplace Equation in Spherical Coordinates

The spherical coordinate system is probably the most useful of all coordinate systems in study of electrostatics, particularly at the microscopic level. In spherical coordinates (r, θ, ϕ), the Laplace equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{3.1}$$

Try separation of variables $\Phi(r, \theta, \phi) = R(r)Y(\theta, \phi)$. Then Eq. 3.1 becomes

$$\frac{1}{R}\left[\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right] = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial Y}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\right] \tag{3.2}$$

Since each side must be equal to a constant, $\lambda = l(l+1)$, we get two equations.

Radial equation

$$\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) - l(l+1)R = 0 \tag{3.3}$$

Introducing $U(r) \equiv rR(r)$, we obtain

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \tag{3.4}$$

The general solution of this equation is

$$U(r) = Ar^{l+1} + Br^{-1} \;\rightarrow\; R(r) = Ar^l + \frac{B}{r^{l+1}} \tag{3.5}$$

Angular equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial Y}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2} = -l(l+1)Y \tag{3.6}$$

We solve Eq. 3.6 using the separation of variable method again: $Y(\theta, \phi) = P(\theta)Q(\phi)$. Then we obtain two ordinary differential equations for θ and ϕ :

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \tag{3.7}$$

$$\frac{\partial^2 Q}{\partial\phi^2}+m^2Q=0 \tag{3.8}$$

Equation 3.8 has solutions

$$Q_m(\phi)=e^{\pm im\phi} \tag{3.9}$$

If the full azimuthal range $(0\le\phi\le 2\pi)$ is allowed, $Q(\phi)=Q(\phi+2\pi)$ so that m must be an integer. The functions Q_m form a complete set of orthogonal functions on the interval $0\le\phi\le 2\pi$, which is nothing but a basis for Fourier series. The orthogonality relation is

$$\int_0^{2\pi}Q_m(\phi)Q_{m'}^*(\phi)\,d\phi=\int_0^{2\pi}e^{i(m-m')}\,d\phi=2\pi\,\delta_{mm'} \tag{3.10}$$

Simultaneously, the completeness relation is

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}Q_m(\phi)Q_m^*(\phi')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}=\delta(\phi-\phi') \tag{3.11}$$

Associated Legendre Polynomials

Equation 3.7 can be written in terms of $x\equiv\cos\theta$:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P(x)=0 \tag{3.12}$$

This is the differential equation for the associated Legendre polynomials. Physically acceptable solution (i.e., $-1\le x\le 1$) is obtained only if l is a positive integer or 0. The solution is then associated Legendre polynomial $P_l^m(x)$ where $l=0,1,2,\ldots$ and $-l\le m\le l$.

$P_l^m(x)$ has the form, $(1-x^2)^{\frac{|m|}{2}}$ times a polynomial of order $l-|m|$, which can be obtained by Rodrigues formula,

$$P_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{|m|}{2}}\frac{d^{m+l}}{dx^{m+l}}(x^2-1)^l\quad\text{for}\quad x\ge 0 \tag{3.13}$$

and the parity relation

$$P_l^m(-x)=(-1)^{l-|m|}P_l^m(x) \tag{3.14}$$

The orthogonal relation of the functions P_l^m is expressed as

$$\int_{-1}^{1} P_l^m(x) P_{l'}^m(x)\, dx = \left(\frac{2}{2l+1}\right) \frac{(l+|m|)!}{(l-|m|)!}\, \delta_{ll'} \tag{3.15}$$

The functions $P_l(x)$ for $m = 0$ are called Legendre polynomials which are the solutions when the problem has azimuthal symmetry so that Φ is independent of ϕ. The first few P_l are

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ \dots \tag{3.16}$$

The Legendre polynomials form a complete orthogonal set of functions on the interval $-1 \leq x \leq 1$. The orthogonality condition can be written as

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \frac{2}{2l+1} \delta_{ll'} \tag{3.17}$$

and the completeness relation is expressed as

$$\sum_{l=0}^{\infty} P_l(x) P_l(x') = \frac{2}{2l+1} \delta(x - x') \tag{3.18}$$

Spherical Harmonics

The angular function can be written as $Y_{lm}(\theta, \phi) = N_{lm} P_l^m(\cos\theta) e^{im\phi}$, where N_{lm} is normalization constant. The normalized angular functions,

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_{lm}(\cos\theta) e^{im\phi} \quad \text{for } m \geq 0 \tag{3.19}$$

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi) \tag{3.20}$$

are called spherical harmonics.

Y_{lm} is either even or odd, depending on l, i.e., $Y_{lm}(-\mathbf{x}) = (-1)^l Y_{lm}(\mathbf{x})$:

In the spherical coordinate, $\mathbf{x} \to -\mathbf{x}$ corresponds to $r \to r,\ \theta \to \pi - \theta, \phi \to \pi + \phi$, therefore

$$\begin{aligned} Y_{lm}(\pi - \theta, \pi + \phi) &= \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_{lm}(\cos(\pi - \theta)) e^{im(\phi+\pi)} \\ &= \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}}\, (-1)^{l-m} P_{lm}(\cos\theta)(-1)^m e^{i\phi} \\ &= (-1)^l Y_{lm}(\theta, \phi) \end{aligned} \tag{3.21}$$

The first few spherical harmonics are

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta, \ \ldots. \tag{3.22}$$

Spherical harmonics form an orthonormal basis for θ and ϕ:

$$\int Y^*_{lm}(\theta,\phi)\, Y_{l'm'}(\theta,\phi)\, d\Omega = \delta_{ll'}\delta_{mm'} \tag{3.23}$$

The corresponding completeness relation is written as

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta') \tag{3.24}$$

General Solution

The general solution for a boundary-value problem in spherical coordinates can be written as

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)}\right] Y_{lm}(\theta,\phi) \tag{3.25}$$

3.2 Boundary-Value Problems with Azimuthal Symmetry

We consider physical situations with complete rotational symmetry about the z-axis (azimuthal symmetry or axial symmetry). This means that the general solution is independent of ϕ, i.e., $m = 0$ in Eq. 3.25:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{3.26}$$

The coefficients A_l and B_l can be determined by the boundary conditions.

Spherical Shell

Suppose that the potential $V(\theta)$ is specified on the surface of a spherical shell of radius a. Inside the shell, $B_l = 0$ for all l because the potential at origin must be finite. The boundary condition at $r = a$ leads to

$$V(\theta) = \sum_{l=0}^{\infty} A_l a^l\, P_l(\cos\theta) \tag{3.27}$$

Using the orthogonality relation Eq. 3.17, we can evaluate the coefficients A_l,

$$A_l = \frac{2l+1}{2a^l}\int_0^{\pi} V(\theta) P_l(\cos\theta)\sin\theta\, d\theta \tag{3.28}$$

On the other hand, outside the shell, $A_l = 0$ for all l because the potential for $r \to \infty$ must be finite and the boundary condition gives rise to

$$B_l = \frac{2l+1}{2} a^{l+1} \int_0^\pi V(\theta) P_l(\cos\theta) \sin\theta \, d\theta \tag{3.29}$$

Two hemispheres at equal and opposite potentials

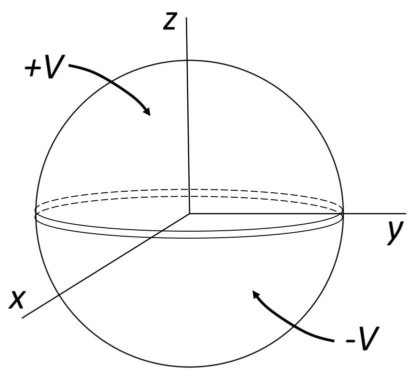

Fig 3.1. Conducting sphere of radius *a* made up of two hemispherical shells separated by a thin insulating ring. They are kept at different potentials, +*V* and −*V*.

Consider a conducting sphere composed of two hemispheres at equal and opposite potentials as shown in Fig. 3.1. Then, inside the sphere, the coefficients A_l are

$$\begin{aligned} A_l &= \tfrac{2l+1}{2a^l} V \left[\int_0^1 P_l(x)\, dx - \int_{-1}^0 P_l(x)\, dx \right] \\ &= \tfrac{2l+1}{2a^l} V \left[\int_0^1 P_l(x)\, dx - \int_0^1 P_l(-x)\, dx \right] \\ &= \tfrac{2l+1}{2a^l} V [1 - (-1)^l] \int_0^1 P_l(x)\, dx \end{aligned}$$

Thus, for even l, $A_l = 0$, and for odd l, using Rodrigues formula, we obtain

$$A_l = \frac{2l+1}{a^l} V \int_0^1 P_l(x)\, dx = \left(-\frac{1}{2}\right)^{\frac{l-1}{2}} \frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!} \frac{V}{a^l} \tag{3.30}$$

Similarly, outside the sphere, for even l, $B_l = 0$, and for odd l

$$B_l = a^{2l+1} A_l \tag{3.31}$$

Using Eq. 3.30 and 3.31, we obtain the potential in entire space:

$$\Phi(r,\theta) = V \begin{cases} \frac{3}{2}\left(\frac{r}{a}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) \cdots, & \text{for } r<a \\ \frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) \cdots, & \text{for } r>a \end{cases} \tag{3.32}$$

Metal sphere in a uniform electric field

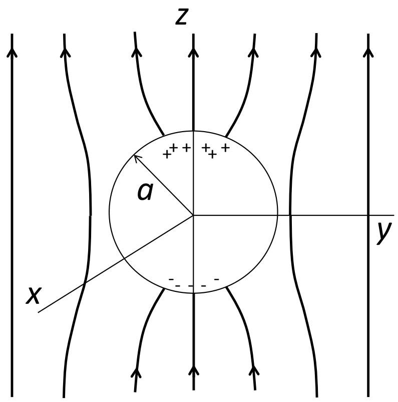

Fig 3.2.

An uncharged metal sphere of radius a is placed in an otherwise uniform electric field $\mathbf{E} = E_0\mathbf{e}_z$ as shown in Fig. 3.2. The potential is given by

$$\Phi(r,\theta) \cong -E_0 z = -E_0 r\cos\theta \tag{3.33}$$

at large distances from the ball, where $\Phi = 0$ in the equatorial plane at $z = 0$. Accordingly, the boundary condition at the surface is $\Phi(a,\theta) = 0$. Referring to the general solution Eq. 3.26, we can immediately set all A_l except for $A_1(= -E_0)$ equal to zero. At $r = a$, we have

$$\Phi(a,\theta) = 0 = -E_0 a\cos\theta + \sum_{l=0}^{\infty}\frac{B_l}{a^{l+1}}\,P_l(\cos\theta) \tag{3.34}$$

We can determine B_l from this equation: $B_1 = E_0 a^3$ and $B_l = 0$ for $l \neq 1$. We finally have

$$\Phi(r,\theta) = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \tag{3.35}$$

The (normal) electric field at the surface is

$$E_r = -\left(\frac{\partial\Phi}{\partial r}\right)_{r=a} = 3E_0\cos\theta \tag{3.36}$$

The surface charge density is

$$\sigma(\theta) = \varepsilon_0 E_r = 3\varepsilon_0 E_0\cos\theta \tag{3.37}$$

Point charge on the *z*-axis

The potential at **x** due to a unit point charge at **x'** can be expressed as an important expansion:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \tag{3.38}$$

Where $r_<$ and $r_>$ are the smaller and larger of r and r', respectively, and γ is the angle between **x** and **x'**. We can prove this equation by aligning **x'** along the *z*-axis as shown in Fig. 3.3. This system has azimuthal symmetry, thus we can expand the potential as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta) \tag{3.39}$$

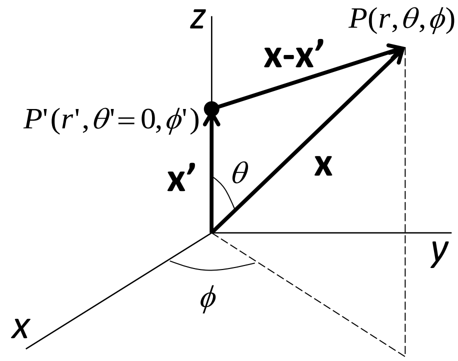

Fig. 3.3

If the point **x** is on the *z*-axis,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{|r-r'|}=\frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l=\sum_{l=0}^{\infty}\left[A_l r^l+B_l r^{-(l+1)}\right] \tag{3.40}$$

Therefore, we can write

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) \tag{3.41}$$

In fact, this equation is independent of coordinate system because θ is the angle between **x** and **x'**, i.e., $\theta=\gamma$. This proves the general result of Eq. 3.38.

Charged circular ring

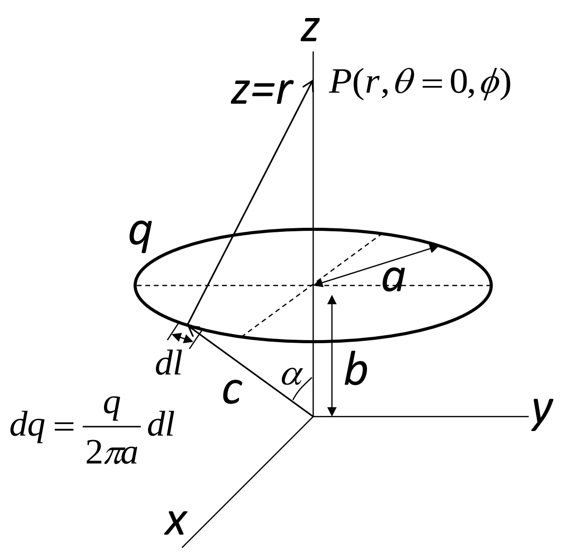

Fig 3.4. Charged ring of radius a and total charge of q located on the *z*-axis with center at $z=b$.

The potential on the *z*-axis due to a charged ring shown in Fig. 3.4 is

$$\Phi(z=r)=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{(r^2+c^2-2cr\cos\alpha)^{1/2}}=\frac{q}{4\pi\varepsilon_0}\frac{1}{(r^2+c^2-2cr\cos\alpha)^{1/2}} \tag{3.42}$$

where $c^2=a^2+b^2$ and $\tan\alpha=\frac{a}{b}$. We can expand Eq. 3.42 using Eq. 3.38.

$$\Phi(z=r)=\frac{q}{4\pi\varepsilon_0}\begin{cases}\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}}P_l(\cos\alpha), & \text{for } r>c\\ \sum_{l=0}^{\infty}\frac{r^l}{c^{l+1}}P_l(\cos\alpha), & \text{for } r<c\end{cases} \tag{3.43}$$

Thus the potential at any point in space is

$$\Phi(r,\theta)=\frac{q}{4\pi\varepsilon_0}\begin{cases}\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}}P_l(\cos\alpha)P_l(\cos\theta), & \text{for } r>c\\ \sum_{l=0}^{\infty}\frac{r^l}{c^{l+1}}P_l(\cos\alpha)\,P_l(\cos\theta), & \text{for } r<c\end{cases} \tag{3.44}$$

3.3 Electric Fields Near a Sharp Point of Conductor

We discuss how electric fields behave near a sharp point of conductor. We consider a conical conducting tip which possesses azimuthal symmetry as shown in Fig. 3.5.

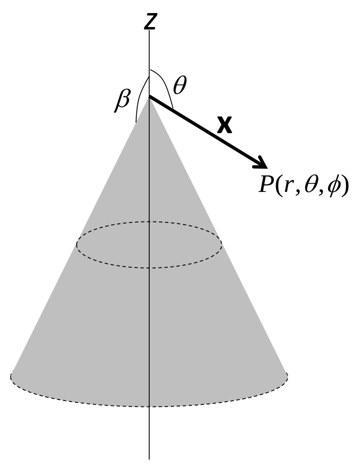

Fig 3.5. Conical conducting tip of opening angle β located at the origin

The basic solution to the Laplace boundary-value problem of Fig. 3.5 is $Ar^{\nu}P_{\nu}(\cos\theta)$, where the order parameter ν is determined by the opening angle β. Since the potential must vanish at $\theta=\beta$,

$$P_{\nu}(\cos\beta)=0 \tag{3.45}$$

The general solution for the potential is expressed as

$$\Phi(r,\theta)=\sum_{k=0}^{\infty}A_k r^{\nu_k}P_{\nu_k}(\cos\theta) \tag{3.46}$$

The potential near the tip is approximately

$$\Phi(r,\theta) \cong A_0 r^{\nu_0} P_{\nu_0}(\cos\theta) \tag{3.47}$$

where ν_0 is the smallest root of Eq. 3.45. The components of electric field and the surface-charge density near the tip are

$$\begin{cases} E_r = -\dfrac{\partial\Phi}{\partial r} \cong -\nu_0 A_0 r^{\nu_0-1} P_{\nu_0}(\cos\theta) \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial\Phi}{\partial\theta} \cong A_0 r^{\nu_0-1} \sin\theta\, P'_{\nu_0}(\cos\theta) \\ \sigma(r) = -\varepsilon_0 E_\theta|_{\theta=\beta} \cong -\varepsilon_0 A_0 r^{\nu_0-1} \sin\beta\, P'_{\nu_0}(\cos\beta) \end{cases} \tag{3.48}$$

The field and charge density all vary as r^{ν_0-1} as $r \to 0$, therefore they are singular at $r = 0$ for $\nu_0 < 1$. When the tip is sharp, i.e., $\beta \to \pi$, an approximate expression for ν_0 as a function β is

$$\nu_0 \cong \left[2\ln\left(\frac{2}{\pi-\beta}\right)\right]^{-1} \tag{3.49}$$

Figure 3.6 show $\nu_0$as a function of β for $150° < \beta < 180°$. This indicates that the fields near a sharp point of conductor vary as $r^{-1+\epsilon}$ where $\epsilon \ll 1$.

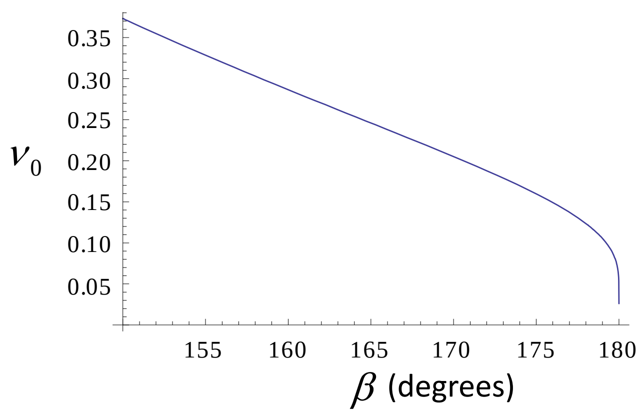

Fig 3.6. The order parameter ν_0 as a function of the opening angle β for $\beta \to \pi$

3.4 Laplace Equation in Cylindrical Coordinates

In cylindrical coordinates (ρ, ϕ, z), the Laplace equation takes the form:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{3.50}$$

Separating the variables by making the substitution

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z) \tag{3.51}$$

Then we obtain the three ordinary differential equations:

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \tag{3.52}$$

$$\frac{d^2Q}{d\phi^2} + \nu^2 Q = 0 \tag{3.53}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \tag{3.54}$$

The solutions of the first two equations are easily obtained:

$$Z(z) = e^{\pm kz} \text{ and } Q(\phi) = e^{\pm im\phi} \tag{3.55}$$

When the full azimuthal angle is allowed, i.e., $0 \leq \phi \leq 2\pi$, m must be an integer. We rewrite the radial equation (Eq. 3.54) by chaning the variable $x = k\rho$.

$$\frac{dR}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{m^2}{x^2}\right) R = 0 \tag{3.56}$$

The solutioins to this equation are best rexpressed as a power series in x. There are two independet solutions, $J_m(x)$ and $N_m(x)$, called Bessel functions of the first kind and Neumann functions, respectively. The Bessel function is defined as

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(m+j)!} \left(\frac{x}{2}\right)^{2j} \tag{3.57}$$

The limiting forms of $J_m(x)$ and $N_m(x)$ for small and large x are usuful to analyze the physical properties of the given bounary-value problem.

For $x \ll 1$

$$J_m(x) \cong \frac{1}{m!}\left(\frac{x}{2}\right)^m \tag{3.58}$$

$$N_m(x) \cong \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & m = 0 \\ -\frac{(m-1)!}{\pi}\left(\frac{2}{x}\right)^m, & m \neq 0 \end{cases} \tag{3.59}$$

For $x \gg 1$

$$J_m(x) \cong \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{3.60}$$

$$N_m(x) \cong \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{3.61}$$

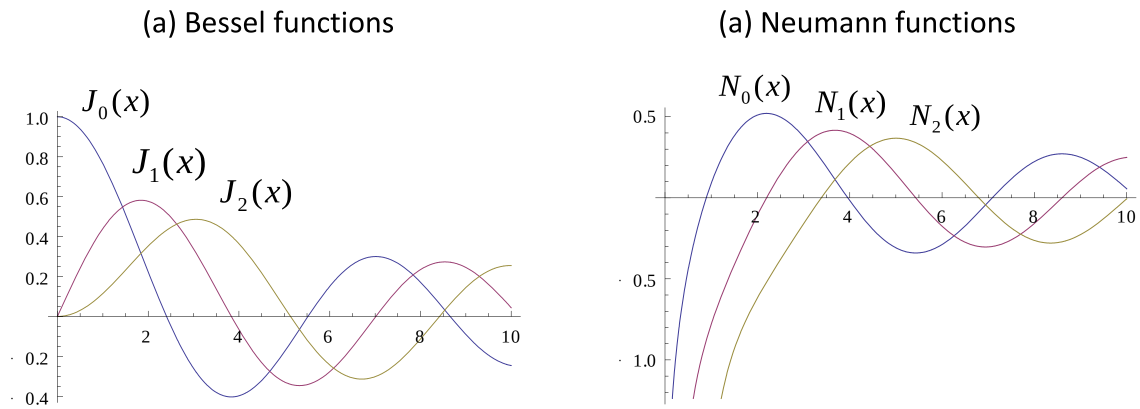

Fig. 3.7. First few Bessel functiosn and Neumann functions

The roots of Bessel functions are important for many boundary-value problems:

$$J_m(x_{mn}) = 0, \quad n = 1,2,3, \dots \tag{3.62}$$

x_{mn} is the nth root of $J_m(x)$:

$$\begin{aligned} m = 0, \quad x_{0n} &= 2.405, 5.520, 8.654, \dots \\ m = 1, \quad x_{1n} &= 3.832, 7.016, 10.173, \dots \\ m = 2, \quad x_{2n} &= 5.136, 8.417, 11.620, \dots \end{aligned}$$

<u>Orthogonal complete set of functions</u>

$\sqrt{\rho}\, J_m\left(x_{mn}\frac{\rho}{a}\right)$ for fixed m and $n = 1,2,3, \dots$ form an orthogonal set on the interval $0 \le \rho \le a$. The normalization integral is

$$\int_0^a \rho J_m\left(x_{mn'}\frac{\rho}{a}\right) J_m\left(x_{mn}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2}\left[J_{m+1}(x_{mn})\right]^2 \delta_{nn'} \tag{3.63}$$

Cylindrical Cavity

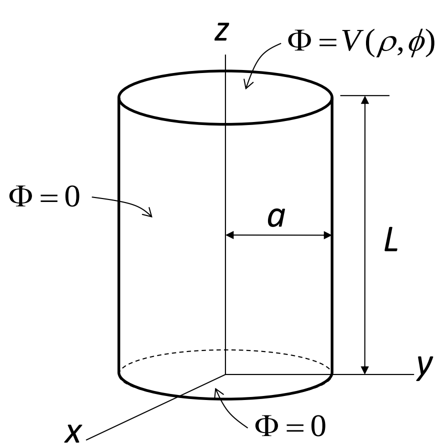

Fig 3.8.

We consider a cylinder of radius a and height L as shown in Fig. 3.8. The potential on the side and the bottom is zero, while the top has a potential $\Phi = V(\rho, \phi)$. We want to find the potential at any point inside the cylinder.

<u>Boundary conditions</u>

- $\Phi = 0$ at $z = 0$ leads to $Z(z) = \sinh kz$.
- $\Phi = 0$ at $\rho = a$, i.e., $J_m(k_{mn}a) = 0$, therefore, $k_{mn} = \frac{x_{mn}}{a}$ (see, Eq. 3.62)

Then, the general solution can be expressed as

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\,(A_{mn}\sin m\phi + B_{mn}\cos m\phi) \tag{3.64}$$

At $z = L$, we have $\Phi = V(\rho,\phi)$, thus

$$\mathrm{V}(\rho,\phi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sinh(k_{mn}L)\,J_m(k_{mn}\rho)(A_{mn}\sin m\phi + B_{mn}\cos m\phi) \tag{3.65}$$

We can determine the coefficients A_{mn} and B_{mn} using the orthogol relations of the sinusoidal and Bessel functions (see Eq. 3.63).

$$\begin{Bmatrix} A_{mn} \\ B_{mn} \end{Bmatrix} = \frac{2\,\mathrm{csch}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\phi \int_0^a \rho d\rho\, V(\rho,\phi) J_m(k_{mn}\rho) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} \tag{3.66}$$

3.5 Poisson Equation and Green Functions in Spherical Coordinates

Addition thorem for spherical harmonics

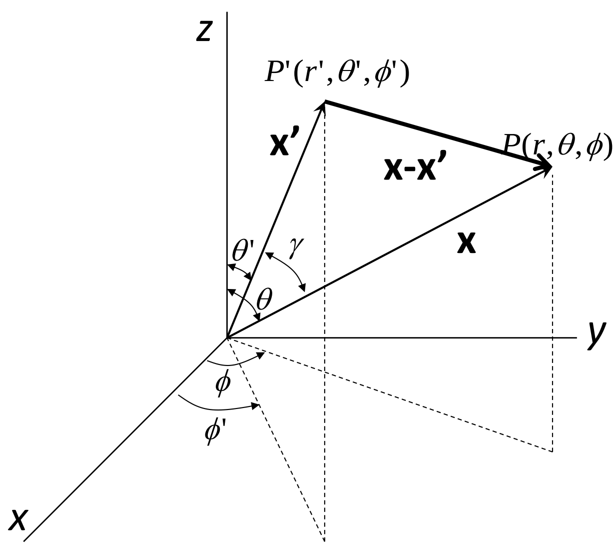

Fig 3.9.

The potential at **x** (**x'**) due to a unit point charge at **x'** (**x**) is an exceedingly important physical quantity in electrostatics. When the two coordinate vectors **x** and **x'** have an angle γ between them, it can be expressed as an important expansion (see Eq. 3.38. We proved this equation in Section 3.2.):

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{3.38}$$

where $r_<$ and $r_>$ are the smaller and larger of r and r', respectively. It is of great interest and use to express this equation in spherical coordiantes. The addition theorem of spherical harmonics is an useful mathematical result for this purpose.

The addition theorem expresses a Legendre polynomial of order l in the angle γ In spherical coordiates as shown in Fig. 3.9:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{3.67}$$

Combining Eq. 3.32 and 3.67, we obtain a completely factorized form in the coordinates **x** and **x'**.

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{3.68}$$

We immediately see that Eq. 3.68 is the expansion of the Green function in spherical coordinates for the case of no boundary surfaces, except at infinity.

Green function with a spherical boundary

The Green function appropriate for Dirichlet boundary conditions on the sphere of radius *a* satisfies the equation (see Eq. 1.27)

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{1.27}$$

and is expressed as (see Eq. 2.13 and 2.14)

$$G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x}) = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}') \tag{2.13}$$

where $\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0$ for $r' > a$ and $G(\mathbf{x}, \mathbf{x}') = 0$ for $r = r' = a$. The discussion of the conducting sphere with the method of images indicates that the Green function can take the form

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{r'\left|\mathbf{x} - \frac{a^2}{r'^2}\mathbf{x}'\right|} \tag{2.14}$$

Using Eq. 3.68 we rewrite Eq. 2.14 as

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right] \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{3.69}$$

The radial factors inside and outside the sphere can be separately expressed as

$$\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1} = \begin{cases} \dfrac{1}{r'^{l+1}}\left(r^l - \dfrac{a^{2l+1}}{r^{l+1}}\right), & r < r' \\ \dfrac{1}{r^{l+1}}\left(r'^l - \dfrac{a^{2l+1}}{r'^{l+1}}\right), & r > r' \end{cases} \tag{3.70}$$

General structure of Green function in spherical coordinates

The Green function appropriate for Dirichlet boundary conditions satisfies the equation (see Eq. 1.27)

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{1.27}$$

In spherical coordinates the delta function can be written

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{r^2}\delta(r - r')\delta(\phi - \phi')\delta(\cos\theta - \cos\theta') \tag{3.71}$$

Using the completeness relation for spherical harmonics (Eq. 3.24), we obtain

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{r^2}\delta(r - r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi')\, Y_{lm}(\theta, \phi) \tag{3.72}$$

Equation 1.27 and 3.72 leads to the expansion of the Green function

$$G(\mathbf{x} - \mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l(r, r') Y_{lm}^*(\theta', \phi')\, Y_{lm}(\theta, \phi) \tag{3.73}$$

and the equation for the radial Green function

$$\frac{1}{r}\frac{d^2}{dr^2}[r g_l(r, r')] - \frac{l(l+1)}{r^2} g_l(r, r') = -\frac{4\pi}{r^2}\delta(r - r') \tag{3.74}$$

The general solution of this equation for $r \neq r'$ (see Eq. 3.5) can be written as

$$g_l(r, r') = \begin{cases} A(r')r^l + \dfrac{B(r')}{r^{l+1}} & \text{for } r < r' \\ A'(r')r^l + \dfrac{B'(r')}{r^{l+1}} & \text{for } r > r' \end{cases} \tag{3.75}$$

The coefficients *A*, *B*, *A'*, *B'* are functions of *r'* to be determined by the boundary conditions, the discontinuity at $r = r'$, and the symmetry of $g_l(r, r') = g_l(r', r)$.

Green function with noboundary

For the case of no boundary, g_l must be finite for $r \to 0$ and ∞, therefore B and A' are zero.

$$g_l(r,r') = \begin{cases} A(r')r^l & \text{for } r < r' \\ \dfrac{B'(r')}{r^{l+1}} & \text{for } r > r' \end{cases} \tag{3.76}$$

Then, the symmetry of of $g_l(r,r') = g_l(r',r)$ leads to

$$g_l(r,r') = g_l(r',r) = C\frac{r_<^l}{r_>^{l+1}} \tag{3.77}$$

where $r_<$ and $r_>$ are the smaller and larger of r and r', respectively. We determine the constant C using the discontinuity at $r = r'$. Integrating the radial equation 3.74 over the infinitesimally narrow interval from $r = r' - \epsilon$ to $r = r' + \epsilon$ with very small ϵ, we obtain

$$\frac{d}{dr}[rg_l(r,r')]\Big|_{r'-\epsilon}^{r'+\epsilon} = -\frac{4\pi}{r'} \tag{3.78}$$

Substituning Eq. 3.77 into Eq. 3.78, we find

$$-\frac{C(l+1)}{r'} - \frac{Cl}{r'} = -\frac{4\pi}{r'} \quad \to \quad C = \frac{4\pi}{2l+1} \quad \to \quad g_l(r,r') = \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} \tag{3.79}$$

This reduces to Eq. 3.68

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{3.68}$$

Green function with two concentric spheres

Suppose that the boundary surfaces are concentric spheres at $r = a$ and $r = b$. $G(\mathbf{x},\mathbf{x}') = 0$ on the surfaces gives rise to

$$g_l(r,r') = \begin{cases} A\left(r^l - \dfrac{a^{2l+1}}{r^{l+1}}\right) & \text{for } r < r' \\ B'\left(\dfrac{1}{r^{l+1}} - \dfrac{r^l}{b^{2l+1}}\right) & \text{for } r > r' \end{cases} \tag{3.80}$$

The symmetry of of $g_l(r,r') = g_l(r',r)$ requires

$$g_l(r,r') = C\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \tag{3.81}$$

Using the discontinuity at $r = r'$ (Eq. 3.78), we obtain the constant C:

$$C = \frac{4\pi}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} \tag{3.82}$$

Combining Eq. 3.73, 3.81, and 3.82, we find the expansion of the Green function

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

(3.83)

Charged ring inside a grounded sphere

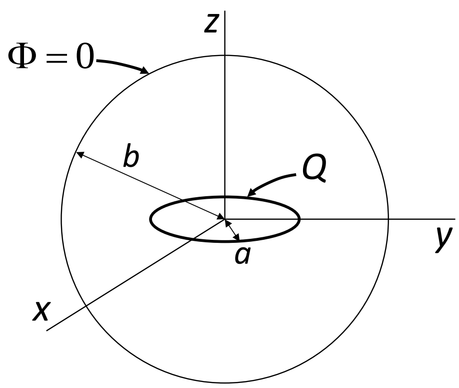

Fig 3.10. Charged ring of radius a and total charge Q inside a grounded, conducting sphere of radius b.

We consider a spherical cavity of radius b with a concnetric righ of charge of radius a and total charge Q as shown in Fig. 3.10. The charge density can be written as

$$\rho(\mathbf{x}') = \frac{Q}{2\pi a^2}\delta(r'-a)\delta(\cos\theta') \tag{3.84}$$

The general solution of the Poisson equation (Eq. 1.31) is given as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')\,G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\int_S \left[\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right] da' \tag{1.31}$$

Since $\Phi = 0$ on the sphere

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')\,G(\mathbf{x},\mathbf{x}')d^3x' \tag{3.84}$$

Using Eq. 3.83 with $a \to 0$,

$$\Phi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)P_l(0)P_l(\cos\theta)$$

$$= \frac{Q}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} \; r_<^{2n} \left(\frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}} \right) P_{2n}(\cos\theta) \qquad (3.85)$$

where $r_<$ and $r_>$ are the smaller and larger of r and a, respectively.

Uniform line charge inside a grounded sphere

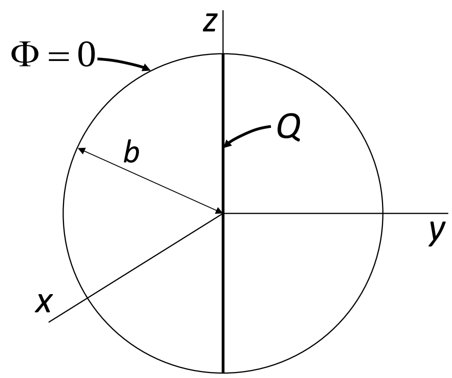

Fig 3.11. Uniform line charge of length $2b$ total charge Q inside a grounded, conducting sphere of radius b.

We consider a spherical cavity of radius b with a uniform line charge of length $2b$ and total charge Q as shown in Fig. 3.11. The charge density can be written as

$$\rho(\mathbf{x}') = \frac{Q}{2b} \frac{1}{2\pi r'^2} \; [\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] \qquad (3.86)$$

Using Eq. 3.84 and $P_l(-1) = (-1)^l$, we obtain the solution

$$\Phi(\mathbf{x}) = \frac{Q}{8\pi\varepsilon_0 b} \sum_{l=0}^{\infty} [1 + (-1)^l] P_l(\cos\theta) \int_0^b r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) dr' \qquad (3.87)$$

The integral must be broken up into two intervals

$$\int_0^b r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) dr' = \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \int_0^r r'^l \, dr' + r^l \int_r^b \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}} \right) dr'$$

$$= \frac{(2l+1)}{l(l+1)} \left[1 - \left(\frac{r}{b} \right)^l \right] \qquad \text{for } l \geq 1 \qquad (3.88)$$

For $l = 0$, this result is indeterminate, and hence we use its limiting behavior

$$\int_0^b \left(\frac{1}{r_>} - \frac{1}{b} \right) dr' = \int_0^r \left(\frac{1}{r} - \frac{1}{b} \right) dr' + \int_r^b \left(\frac{1}{r'} - \frac{1}{b} \right) dr'$$

$$= 1 - \frac{r}{b} + \ln\left(\frac{b}{r} \right) - \left(1 - \frac{r}{b} \right) = \ln\left(\frac{b}{r} \right) \qquad \text{for } l = 0 \qquad (3.89)$$

Substituting Eq. 3.88 and 3.98 into Eq. 3.87, we find

$$\Phi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0 b}\left\{\ln\left(\frac{b}{r}\right) + \sum_{n=1}^{\infty}\frac{(4n+1)}{2n(2n+1)}\left[1-\left(\frac{r}{b}\right)^{2n}\right]P_{2n}(\cos\theta)\right\} \tag{3.90}$$

The surface charge density on the grounded sphere is

$$\sigma(\theta) = \varepsilon_0 \frac{\partial\Phi}{\partial r}\bigg|_{r=b} = -\frac{Q}{4\pi b^2}\left[1+\sum_{n=1}^{\infty}\frac{(4n+1)}{(2n+1)}P_{2n}(\cos\theta)\right] \tag{3.91}$$

3.6 Conducting Plane with a Circular Hole

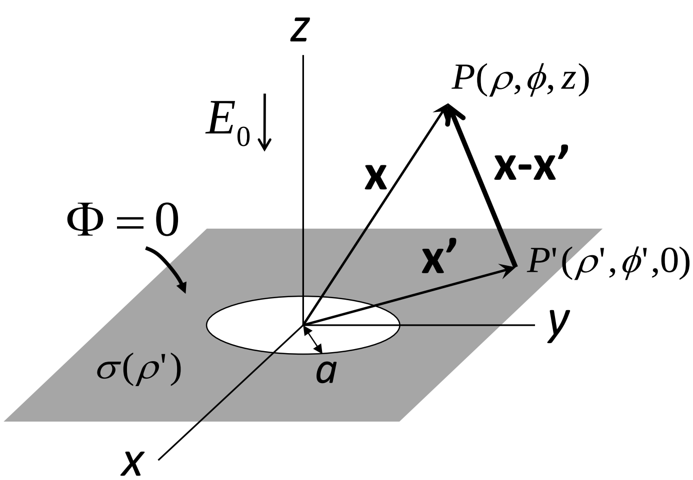

Fig 3.12. A circular hole of radius a is punctured in a grounded conducting plane. The limiting behavior of the electric field far from the hole is

$$\mathbf{E} = \begin{cases} -E_0\mathbf{e}_z, & z>0 \\ 0, & z<0 \end{cases}$$

We consider a grounded conducting plane in which a circular hole is punctured. Figure 3.12 illustrates the geometry. The electric field far from hole has only a *z* component: $E_z = -E_0$ for $z > 0$ and $E_z = 0$ for $z < 0$. From the limiting behavior of the field, we can deduce that the potential takes the form

$$\Phi = \begin{cases} E_0 z + \Phi^{(1)}, & z>0 \\ \Phi^{(1)}, & z<0 \end{cases} \tag{3.92}$$

where $\Phi^{(1)}$ is the potential due to a rearrangement of surface charge near the hole, $\sigma^{(1)}$. Because of the azimuthal symmetry, $\Phi^{(1)}$ is independent of ϕ, and it can be expressed as

$$\Phi^{(1)}(\rho,z) = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma^{(1)}(\rho')\rho' d\rho' d\phi'}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi'+z^2}} \tag{3.93}$$

Mixed boundary conditions

$\Phi^{(1)}$ is even in *z*, and hence $E_z^{(1)}$ is odd. Since the total *z* component of electric field must be continuous across $z = 0$ in the hole, we must have (for $\rho < a$)

$$-E_0 + E_z^{(1)}\Big|_{z=0+} = E_z^{(1)}\Big|_{z=0-} \tag{3.94}$$

Since $E_z^{(1)}$ is odd,

$$E_z^{(1)}\Big|_{z=0+} = -E_z^{(1)}\Big|_{z=0-} = \frac{1}{2}E_0 \tag{3.95}$$

This relation and the ground potential of the conducting surface complete the boundary conditions for the entire *xy* plane. We therefore have mixed boundary conditions

$$\begin{cases} \dfrac{\partial \Phi^{(1)}}{\partial z}\Bigg|_{z=0+} = -\dfrac{1}{2}E_0, & 0 \le \rho < a \\ \Phi^{(1)}\Big|_{z=0} = 0 \quad , & a \le \rho < \infty \end{cases} \tag{3.96}$$

Solution of Laplace equation in cylindrical coordinates

The general solution of the Laplace equation in cylindrical coordiates (see Eq. 3.51, 3.55 and 3.57) is

$$\Phi = R(\rho)Q(\phi)Z(z) = J_m(k\rho)e^{\pm im\phi}e^{\pm kz} \tag{3.97}$$

Because of the azimuthal symmetry, $\Phi^{(1)}$ is independent of ϕ, and hence $m = 0$. Therefore, $\Phi^{(1)}$ can be written in terms of cylindrical coordinates

$$\Phi^{(1)}(\rho, z) = \int_0^\infty A(k)e^{-k|z|}J_0(k\rho)\,dk \tag{3.98}$$

Boundary conditions and integral equatoins

The boundary conditions (Eq. 3.96) for the general solution (Eq. 3.98) give rise to the integral equatoins of the coefficient $A(k)$:

$$\begin{cases} \displaystyle\int_0^\infty kA(k)J_0(k\rho)\,dk = \frac{1}{2}E_0, & 0 \le \rho < a \\ \displaystyle\int_0^\infty A(k)J_0(k\rho)\,dk = 0 \quad , & a \le \rho < \infty \end{cases} \tag{3.99}$$

There exists an analytic solution of these interal equations.

$$A(k) = \frac{E_0 a^2}{\pi} j_1(ka) = \frac{E_0 a^2}{\pi}\left(\frac{\sin ka}{k^2 a^2} - \frac{\cos ka}{ka}\right) \tag{3.100}$$

where $j_1(x)$ is the spherical Bessel function of order 1.

Multipole expansion in the far-field region

In the far-feld region, i.e., in the region for $|z|$ and/or $\rho \gg a$, the integral in Eq. 3.98 is mainly determined by the contributions around $k = 0$, more precisely, for $k \ll \frac{1}{a}$. The expansion of $A(k)$ for small ka takes the form

$$A(k) \cong \frac{E_0 a^2}{3\pi}\left[ka - \frac{(ka)^3}{10} + \cdots\right] \tag{3.101}$$

The leading term gives rise to the asymptotic potential

$$\begin{aligned}\Phi^{(1)}(\rho,z) &\cong \frac{E_0 a^3}{3\pi}\int_0^\infty k e^{-k|z|} J_0(k\rho)\,dk \\ &= -\frac{E_0 a^3}{3\pi}\frac{d}{d|z|}\int_0^\infty e^{-k|z|} J_0(k\rho)\,dk \\ &= -\frac{E_0 a^3}{3\pi}\frac{d}{d|z|}\left(\frac{1}{\sqrt{\rho^2+z^2}}\right) = \frac{E_0 a^3}{3\pi}\frac{|z|}{(\rho^2+z^2)^{\frac{3}{2}}}\end{aligned}$$

Here we have the asymptotic potential

$$\Phi^{(1)}(\rho,z) \cong \frac{E_0 a^3}{3\pi}\frac{|z|}{r^3} \tag{3.102}$$

falling off with distance as r^{-2} and having an effective electric dipole moment,

$$\mathbf{p} = \mp\frac{4}{3}\varepsilon_0 \mathbf{E}_0 a^3 \tag{3.103}$$

where – for $z > 0$ and $+$ for $z < 0$.

Potential in the near-field region

The potential in the neighborhood of the hole must be calculated from the exact expression

$$\Phi^{(1)}(\rho,z) = \frac{E_0 a^2}{\pi}\int_0^\infty j_1(ka) e^{-k|z|} J_0(k\rho)\,dk \tag{3.104}$$

An integration by parts and Laplace transforms results in

$$\Phi^{(1)}(\rho,z) = \frac{E_0 a}{\pi}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{|z|}{a}\tan^{-1}\left(\sqrt{\frac{2}{R+\lambda}}\right)\right] \tag{3.105}$$

where $\lambda = \frac{1}{a^2}(z^2+\rho^2-a^2), \quad R = \sqrt{\lambda^2 + \frac{4z^2}{a^2}}$ (3.106)

- For $\rho = 0$,

$$\Phi^{(1)}(0,z) = \frac{E_0 a}{\pi}\left[1 - \frac{|z|}{a}\tan^{-1}\left(\frac{a}{|z|}\right)\right] \tag{3.107}$$

  For $|z| \gg a$,

$$\Phi^{(1)}(0,z) \cong \frac{E_0 a}{\pi}\left\{1 - \frac{|z|}{a}\left[\frac{a}{|z|} - \frac{1}{3}\left(\frac{a}{|z|}\right)^3 + \cdots\right]\right\} = \frac{E_0 a^3}{3\pi}\frac{1}{|z|^2} \tag{3.108}$$

  This is consistent with the dipole approximation in Eq. 3.102.
- In the plane of the opening ($z = 0$),

$$\Phi^{(1)}(\rho,0) = \frac{E_0}{\pi}\sqrt{a^2 - \rho^2} \quad \text{for } \ 0 \le \rho < a \tag{3.109}$$

  The tangential electric field in the opening is a radial field,

$$\mathbf{E}_{\parallel}(\rho,0) = \frac{E_0}{\pi}\frac{\boldsymbol{\rho}}{\sqrt{a^2 - \rho^2}} \tag{3.110}$$

  The boundary condition in Eq. 3.96 indicates the normal component of electric field in the opening,

$$E_z(\rho,0) = -\frac{1}{2}E_0 \tag{3.111}$$

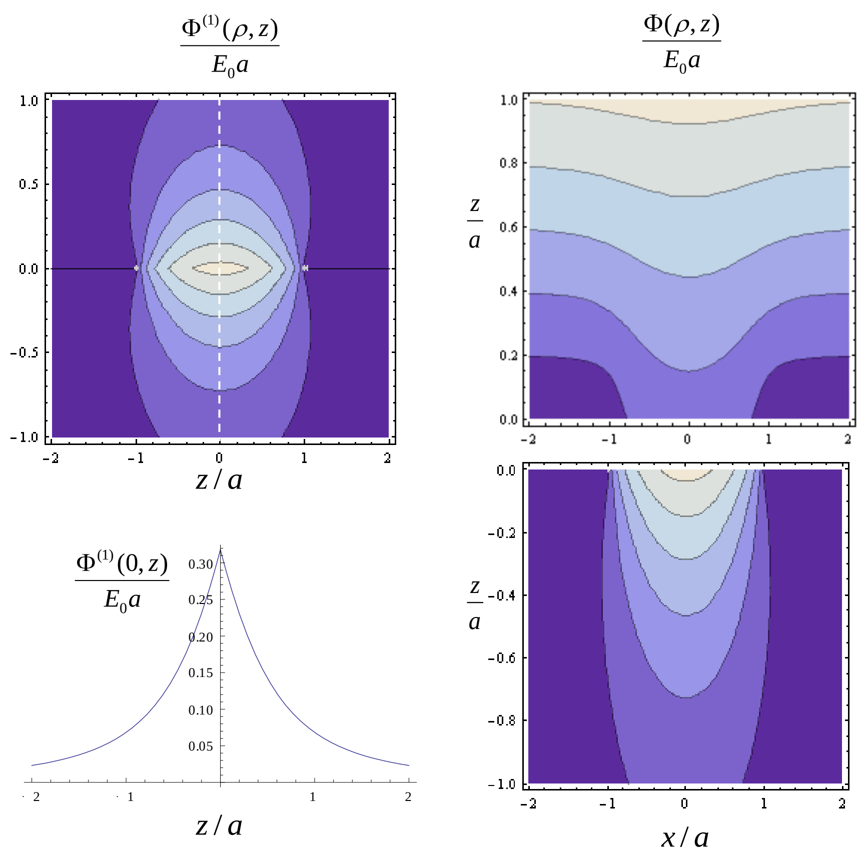

Fig 3.13. Equipotential contours of the additional potential $\Phi^{(1)}$ and the total potential Φ in the vicinity of the hole. $\Phi^{(1)}(0,z)$ as a function of *z* is also shown.