Chapter 1. Introduction to Electrostatics

1.1 Electric charge, Coulomb's Law, and Electric field

Electric charge

Fundamental and characteristic property of the elementary particles

There are two and only two kinds of electric charge known as negative and positive.

Charge conservation: Net charge is conserved in a closed system.

Coulomb's law

- Two point charges exert on each other forces that act along the line joining them and are inversely proportional to the square of the distance between them.
- These forces are proportional to the product of the charges: repulsive for like charges and attractive for unlike charges.

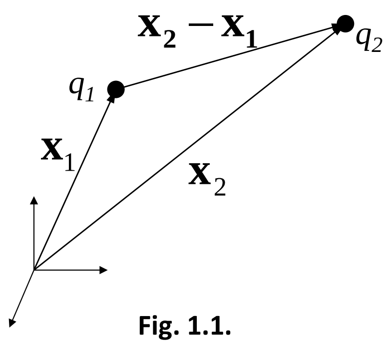


Fig. 1.1.

Force on q_2 due to q_1

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\mathbf{x_2} - \mathbf{x_1}}{|\mathbf{x_2} - \mathbf{x_1}|^3} \qquad (1.1)$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$$

Permitivity of free space

Electric field

The electric field at $\boldsymbol{x}$ due to q_1 is defined as

$$\mathbf{E}(\boldsymbol{x}) = \frac{\text{force on a test charge of infinisimally small charge at } \boldsymbol{x}}{\text{the charge of the test charge}} = \lim_{q \to 0} \frac{\mathbf{F}_q}{q}$$

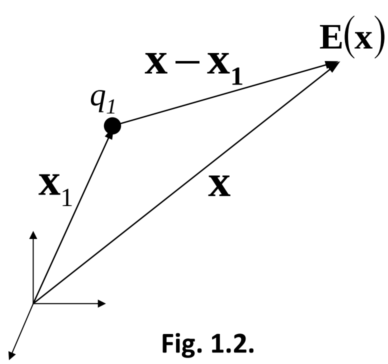


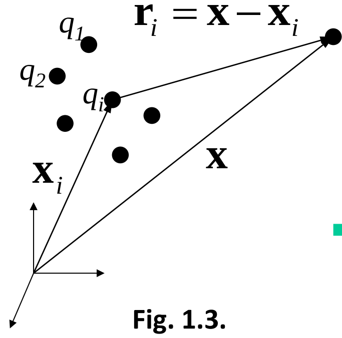
Fig. 1.2.

Electric field at $\mathbf{x}$ due to q_1

$$\mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x_1}}{|\mathbf{x} - \mathbf{x_1}|^3} \qquad (1.2)$$

Linear superposition of electric fields

The total electric field due to many point charges is the vector sum of the electric fields due to the individual charges. If the charge distribution can be described by a charge density $\rho(\mathbf{x}')$, the sum is replaced by an integral.



$$\mathbf{E}(\mathbf{x}) = \mathbf{E}_1(\mathbf{x}) + \mathbf{E}_2(\mathbf{x}) + \cdots$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{x}\text{-}\mathbf{x}_i}{|\mathbf{x}\text{-}\mathbf{x}_i|^3} \qquad (1.3)$$

$$\Rightarrow \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x}\text{-}\mathbf{x}'}{|\mathbf{x}\text{-}\mathbf{x}'|^3} d^3x' \qquad (1.4)$$

Fig. 1.3.

The discrete set of point charges can be described as $\rho(\mathbf{x}') = \sum_{i=1}^{n} q_i \delta(\mathbf{x}' - \mathbf{x}_i)$. Then, the electric field is expressed as

$$E(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0} \int \sum_{i=1}^{n} q_i \delta(\mathbf{x}' - \mathbf{x}_i) \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \int \delta(\mathbf{x}' - \mathbf{x}_i) \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3}$$

1.2 Gauss's Law

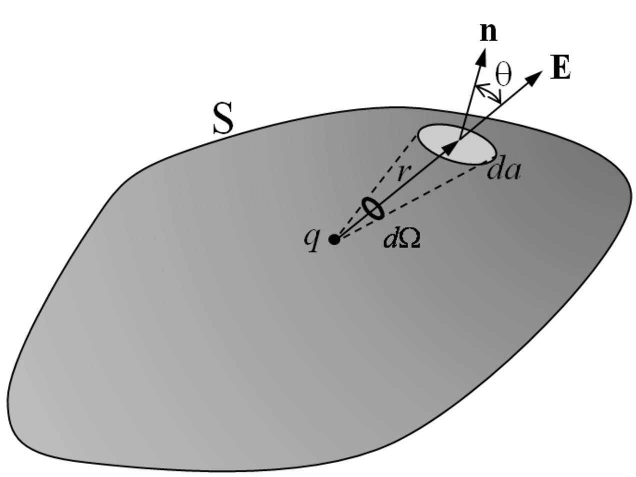


Fig 1.4. An imaginary closed surface S that encloses a point charge *q*

Imagine a closed surface enclosing a point charge q (see Fig. 1.4). The electric field at a point on the surface is $\boldsymbol{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}$, where r is the distance from the charge to the point. Then,

$$\mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}da = \frac{q}{4\pi\varepsilon_0}d\Omega,$$

where **n** is the outwardly directed unit normal to the surface at that point, *da* is an element of surface area, and θ is the angle between **n** and **E**, and $d\Omega$ is the element of solid angle subtended by *da* at the position of the charge. The integration over the whole surface is

$$\oint \mathbf{E}\cdot\mathbf{n}\,da = \begin{cases} q/\varepsilon_0 & \text{If } q \text{ lies inside } S \\ 0 & \text{If } q \text{ lies outside } S \end{cases} \tag{1.5}$$

- Gauss's law for a single point charge

For a continuous charge density $\rho(\mathbf{x})$, Gauss's law becomes:

$$\oint \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x \tag{1.6}$$

where V is the voume enclosed by S.

Applying the divergence theorem, the integration can be written as

$$\int_V \left[\nabla\cdot\mathbf{E} - \frac{\rho(\mathbf{x})}{\varepsilon_0}\right]d^3x = 0 \tag{1.7}$$

This equation must be valid for all volums, that is, for an arbitrary volume V. The validity of this equation implies that

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho(\mathbf{x}) \tag{1.8}$$

- Differential form of Gauss's law

1.3 The Curl of E and the Scalar Potential

The curl of the electric field $\mathbf{E}(\mathbf{x})$ vanishes, i.e.,

$$\nabla\times\mathbf{E}(\mathbf{x}) = 0 \tag{1.9}$$

Proof) The curl of the Eq. 1.4 gives

$$\nabla\times\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\left[\nabla\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right]d^3x'$$

where $\nabla \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \frac{1}{|\mathbf{x}-\mathbf{x}'|^3} \nabla \times (\mathbf{x}-\mathbf{x}') + \left[\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|^3}\right] \times (\mathbf{x}-\mathbf{x}')$

$$= \left[-3\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^5}\right] \times (\mathbf{x}-\mathbf{x}') = 0.$$

Therefore, $\nabla \times \mathbf{E}(\mathbf{x}) = 0.$ (Q.E.D.)

Since $\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)$, the field can be written as

$$\mathbf{E}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0} \nabla \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{1.10}$$

We define the scalar potential (or electrostatic potential) by the equation:

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) \tag{1.11}$$

Then the scalar potential for the charge density $\rho(\mathbf{x}')$ is expressed as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{1.12}$$

The scalar potential is closely related to the potential energy associated with the electrostatic force. The work done in moving a test charge *q* from *A* to *B* is

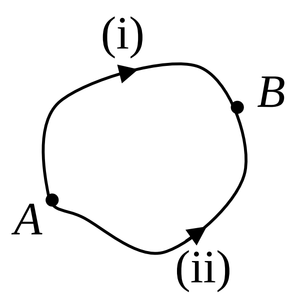


Fig. 1.5.

$$W = -\int_A^B \mathbf{F} \cdot d\mathbf{l} = -q\int_A^B \mathbf{E} \cdot d\mathbf{l} = q\int_A^B \nabla\Phi \cdot d\mathbf{l}$$

$$= q\int_A^B \mathrm{d}\Phi = q(\Phi_B - \Phi_A). \tag{1.13}$$

$q\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic field.

The line integral of the electric field between two points is independent of the path. The line integral vanishes for a closed path:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.14}$$

This is consistent with Stokes's theorem:

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, da = 0 \tag{1.15}$$

where *S* is an open surface bounded by the closed path and ***n*** is the unit vector normal to S.

1.4 Poisson and Laplace Equations

The Gauss's law (Eq. 1.8) and the scalar potential equation (Eq. 1.11) can be combined into one partial differential equation for the scalar potential:

$$\nabla^2\Phi(\mathbf{x}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{x}) \tag{1.16}$$

→ <u>Poisson equation</u>

In regions of no charge,

$$\nabla^2\Phi(\mathbf{x}) = 0 \tag{1.17}$$

→ <u>Laplace equation</u>

We already have a solution for the scalar potential (Eq. 1.12). We can verify that it satisfies the Poisson equation.

$$\nabla^2\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\left[\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]d^3x'$$

(1.18)

Using $\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}')$

$$\nabla^2\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')[-4\pi\delta(\mathbf{x}-\mathbf{x}')]d^3x'$$

$$= -\frac{1}{\varepsilon_0}\rho(\mathbf{x})$$

1.4 Surface Distributions of Charges and Discontinuities in the Electric Field

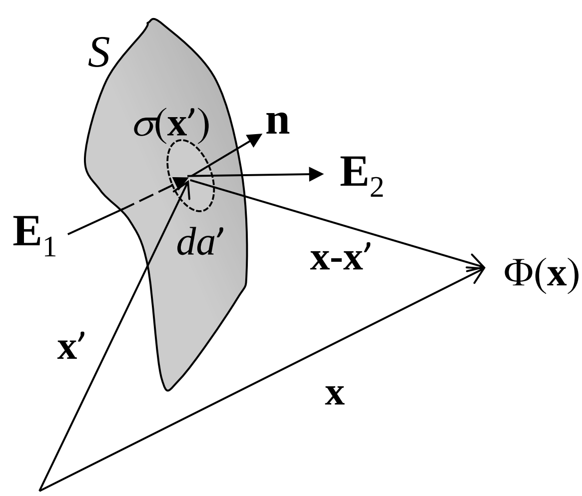


Fig. 1.6. Surface charge distribution σ(x')

The scalar potential due to the surface charge distribution can be expressed as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,da' \tag{1.19}$$

The potential is everyhwere continuous because the electric field is bounded. The electric field, however, is discontinous at the surface. Applying Gauss's law to the small pillbox-shaped surface *da*', we can obtain the following boundary condition:

$$(\mathbf{E}_2 - \mathbf{E}_1)\cdot\mathbf{n} = \sigma/\varepsilon_0 \tag{1.20}$$

1.6 Green's Theorem and Boundary Value Problem

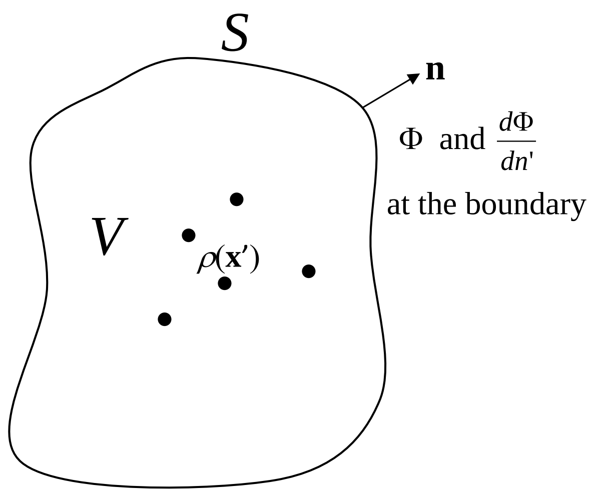


Fig 1.7. Finite region with or without charge inside and with prescribed boundary conditions

If the divergence theorem is applied to the vector field $\boldsymbol{A} = \phi\boldsymbol{\nabla}\psi$, where ϕ and ψ are arbitrary scalar fields,

$$\int_V \nabla\cdot\ (\phi\boldsymbol{\nabla}\psi)\,d^3x = \int_S(\phi\boldsymbol{\nabla}\psi)\cdot\mathbf{n}\,da$$

$$\int_V(\phi\boldsymbol{\nabla}^2\psi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi)\,d^3x = \int_S \phi\frac{\partial\psi}{\partial n}da \tag{1.21}$$

→ <u>Green's first identity</u>

Subtracting the equation with ϕ and ψ interchanged from Eq. 1.21, we obtatin

$$\int_V(\phi\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\phi)\,d^3x = \int_S\left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]da \tag{1.22}$$

→ <u>Green's theorem</u>

If we choose $\psi = \frac{1}{R} \equiv \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ and $\phi = \Phi$ (scalar potential), where **x** is the observation point and **x**' is the integration variable, and use the Poisson equation, Eq. 1.22 becomes

$$\int_V \left[-4\pi\Phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{\varepsilon_0 R}\rho(\mathbf{x}')\right] d^3x' = \int_S \left[\Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \Phi}{\partial n'}\right] da' \qquad (1.23)$$

If the observation point **x** lies within the volume *V*, we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' + \frac{1}{4\pi}\int_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da' \qquad (1.24)$$

- $\Phi(\mathbf{x}) \to \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ when $R \to \infty$.
- If $\rho(\mathbf{x}') = 0$, the potential inside the volume *V* is determined only by the boundary condition, i.e., the values of Φ and $\partial\Phi/\partial n'$ on the surface *S*.

Dirichlet boundary condition – Φ everywhere on the surface *S*

Neumann boundary condition – $\partial\Phi/\partial n'$ on the surface *S*

Uniqueness theorem: The solution of the Poisson equation inside *V* is unique if either Dirichlet or Neumann boundary condition on *S* is satisfied.

Proof) We suppose that two solutions Φ_1 and Φ_2 satisfy the same boundary conditions. Let

$$U = \Phi_2 - \Phi_1 \qquad (1.25)$$

Then, $\nabla^2 U = 0$ insider *V*, and $U = 0$ or $\frac{\partial U}{\partial n} = 0$ on *S*. From Eq. 1.21 with $\phi = \psi = U$, we find

$$\int_V (U\nabla^2 U + \nabla U \cdot \nabla U)\, d^3x = \int_S U\frac{\partial U}{\partial n} da \qquad (1.26)$$

This reduces to $\int_V |\nabla U|^2\, d^3x = 0$ which implies $\nabla U = 0$. Consequently, inside *V*, *U* is constant.

(Q.E.D.)

Formal Solution of boundary value problem with Green function

Green functions satisfy $\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ (1.27)

where $G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}')$ (1.28)

and $\nabla'^2 F(\mathbf{x},\mathbf{x}') = 0$ inside *V*. (1.29)

$F(\mathbf{x},\mathbf{x}')/4\pi\varepsilon_0$ represents the potential due to charges external to the volume *V*.

Applying Green's theorem (Eq. 1.22) with $\phi = \Phi$ and $\psi = G(\mathbf{x},\mathbf{x}')$, we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')\, G(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\int_S \left[G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right] da' \quad (1.30)$$

- For Dirichlet boundary conditions, $G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}'$ on S, then

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')\, G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\int_S \left[\Phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}\right] da' \quad (1.31)$$

- For Neumann boundary conditions, to be consistent with Gauss's theorem,

$$\frac{\partial G_N(\mathbf{x},\mathbf{x}')}{\partial n'} = -\frac{4\pi}{S} \text{ for } \mathbf{x}' \text{ on } S \quad (1.32)$$

$$\Phi(\mathbf{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')\, G_N(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\int_S \left[\frac{\partial\Phi}{\partial n'} G_N\right] da' \quad (1.33)$$

1.7 Electrostatic Potential Energy

Potential energy of a group of point charges

The electrostatic potential energy of a group of n point charges can be obtained by calculating the work to assemble the charges bringing in one at a time:

$$W_1 = 0,\ W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_1}{x_{21}},\ W_3 = \frac{q_3}{4\pi\varepsilon_0}\left(\frac{q_1}{x_{31}} + \frac{q_2}{x_{32}}\right), \ldots,\ W_i = \frac{q_i}{4\pi\varepsilon_0}\sum_{j=1}^{i-1}\frac{q_j}{x_{ij}},\ \ldots,\ W_n$$

where $x_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$. Then, the total potential energy is

$$W = \sum_{i=1}^{n} W_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\frac{q_i q_j}{x_{ij}} = \frac{1}{8\pi\varepsilon_0}\sum_{i(\neq j)=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{x_{ij}} \quad (1.34)$$

The infinite self-energy terms (*i*=*j* terms) are omitted in the double sum.

Potential energy for a continuous charge distribution

$$W = \frac{1}{8\pi\varepsilon_0}\iint \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x d^3x' \quad (1.35)$$

$$= \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x}) d^3x \quad (1.36)$$

Using the Poisson equation and integration by parts, we obtain

$$W = -\frac{\varepsilon_0}{2}\int \Phi\nabla^2\Phi d^3x = \frac{\varepsilon_0}{2}\int |\nabla\Phi|^2 d^3x = \frac{\varepsilon_0}{2}\int |\mathbf{E}|^2 d^3x \quad (1.37)$$

This leads to an energy density, $w = \frac{1}{2}\varepsilon_0|\mathbf{E}|^2$ (1.38)

The energy density is positive definite because Eq. 1.37 contains self-energy.

1.8 Conductors and Electrostatic Energy

Conductors are substances containing large numbers of free charge carriers. The basic electrostatic properties of ideal conductors are

- $\mathbf{E} = 0$ inside a conductor
- $\rho = 0$ inside a conductor
- Any net charge resides on the surface.
- A conductor is an equipotential.
- **E** is perpendicular to the surface, just outside a conductor: $\mathbf{E} = \frac{\sigma}{\varepsilon_0}\mathbf{n}$

Capacitor: Assembly of two conductors that can store equal and opposite charges (±Q)

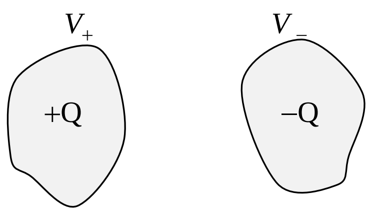


Fig. 1.8. capacitor

Since the potential difference between them is

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \boldsymbol{E} \cdot d\mathbf{l}$$

and **E** is proportional to *Q*, *V* is proportional to *Q*, i.e.,

$$Q = CV \tag{1.39}$$

where *C* is called the capacitance.

The potential energy of the capacitor can be calculated as

$$W = \int_0^Q V\,dq = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2 \tag{1.40}$$

For a system of n conductors, each with potential V_i and charge Q_i, V_i can be written as

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j$$

These equations can be inverted to

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \tag{1.41}$$

where C_{ii} is the capacitance of the *i*-th conductor. The potential energy of the system is

$$W = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} V_i V_j \tag{1.42}$$