

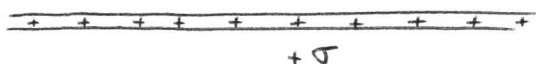
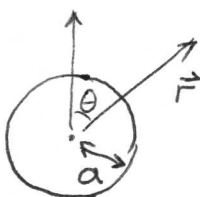
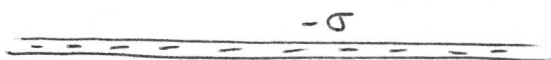
Last time: Boundary conditions at a dielectric interface

$$D_{\perp, \text{above}} = D_{\perp, \text{below}}$$

$$\Phi_{\text{above}} = \Phi_{\text{below}}$$

$$E_{\parallel \text{above}} = E_{\parallel \text{below}}$$

Today, apply these BCs to the problem of a crystal ball inside a capacitor



→ Far away (but still inside capacitor)

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z} = \vec{E}_0$$

$$\begin{aligned} \text{Equivalently } \Phi(\vec{r}) &= -E_0 z \\ &= -E_0 r \cos\theta \end{aligned}$$

Closer to the crystal ball (but still outside the ball) we expect a more complicated potential

$$\Phi_{\text{out}}(\vec{r}) = -E_0 r \cos\theta + \underbrace{\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)}_{\text{No } A_l \text{ terms because } \Phi_{\text{out}} \rightarrow -E_0 r \cos\theta \text{ at large } r.}$$

(2)

Inside the ball

$$\bar{\Phi}_{in}(\vec{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

No B_l terms because $\bar{\Phi}_{in}$ should not diverge at the origin.

At $|\vec{r}| = a$

$$\bar{\Phi}_{out} = \bar{\Phi}_{in}$$

$$-E_0 a \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)$$

Because Legendre polynomials are orthogonal we can compare the coefficient multiplying the l^{th} polynomial.

$$\boxed{l=0} \quad \frac{B_0}{a} = A_0$$

$$\boxed{l=1} \quad -E_0 a + \frac{B_1}{a^2} = A_1 a$$

$$\boxed{l=2} \quad \frac{B_2}{a^3} = A_2 a$$

$$\boxed{l=3} \quad \text{etc...}$$

(3)

Another constraint at the boundary is

$$D_{\perp \text{ above}} = D_{\perp \text{ below}}$$

$$\epsilon_0 \left. \frac{\partial \Phi_{\text{out}}}{\partial r} \right|_{r=a} = \epsilon \left. \frac{\partial \Phi_{\text{in}}}{\partial r} \right|_{r=a}$$

$$\epsilon_0 \left(-E_0 \cos \theta + \sum_{l=0}^{\infty} -(l+1) \frac{B_l}{a^{l+2}} P_l(\cos \theta) \right) = \epsilon \left(\sum_{l=1}^{\infty} A_l l a^{l-1} P_l(\cos \theta) \right)$$

Comparing coefficients

$$\boxed{l=0} \quad -\frac{\epsilon_0 B_0}{a^2} = 0$$

$$\boxed{l=1} \quad -\epsilon_0 E_0 - \frac{\epsilon_0 2B_1}{a^3} = \epsilon A_1$$

$$\boxed{l=2} \quad -\frac{\epsilon_0 3B_2}{a^4} = \epsilon A_2 2a$$

$$\boxed{l=3} \quad \text{etc.}$$

From the two boundary conditions we conclude that only A_1 and B_1 are non-zero.

(4)

At this point it is straight forward to do a page of algebra and find

$$B_1 = \frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} a^3 E_0 = \frac{\chi_e}{\chi_e + 3} a^3 E_0$$

$$A_1 = -\frac{3}{\frac{\epsilon}{\epsilon_0} + 2} E_0 = -\frac{3}{\chi_e + 3} E_0$$

Thus we have determined Φ_{in} and Φ_{out} .

However, there is a way to skip the page of algebra and solve the problem in a way that gives more physical insight:

ALTERNATIVE ENDING

Once we know that only A_1 & B_1 are non-zero:

$$\Phi_{out} = -E_0 z + \underbrace{\frac{B_1 \cos\theta}{r^2}}_{\text{A dipole field}}$$

$$\Phi_{in} = \underbrace{A_1 z}$$

A uniform \vec{E} -field, inside the ball, which implies the polarization \vec{P} must be uniform.