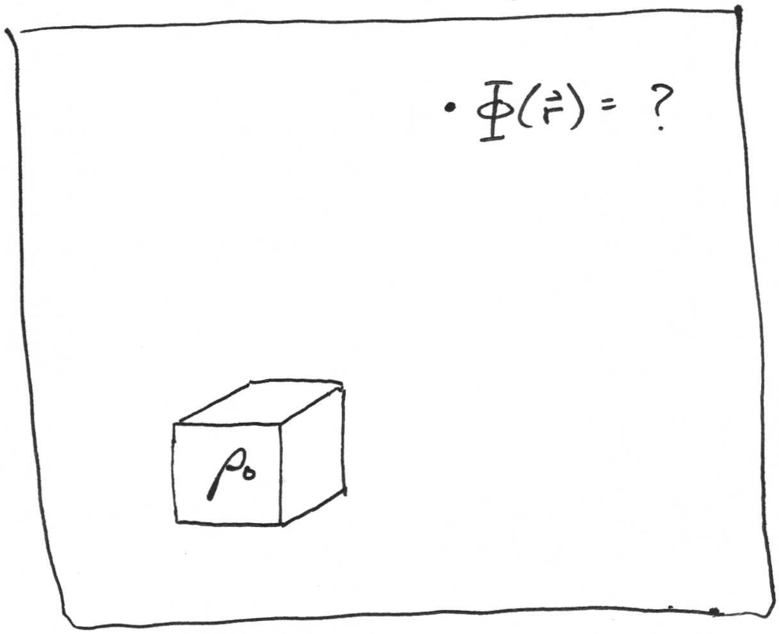
Pop Quiz   Day 15

2015



A cube with edge length a is filled with uniform charge density ρ_0. ~~Give an~~ Write an approximate analytical solution for $\Phi(\vec{r})$ that approaches the true value of $\Phi(\vec{r})$ at large r.

DAY 15

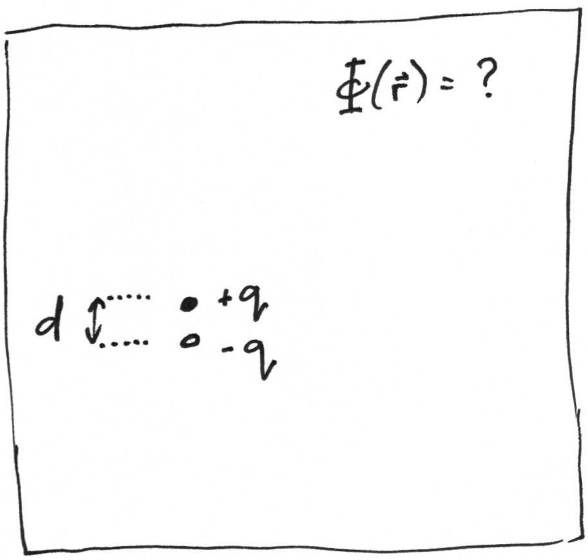
PH 631

2015

Instructor
Ethan Minot

INTRODUCTION TO MULTIPOLE EXPANSIONS

Consider a similar question as the pop quiz:



Find an analytical expression, $\Phi_{approx}(r,\theta)$ that approaches the true value of $\Phi(\vec{r})$ at large $\vec{r}$.

Note: $Q_{Tot} = 0$, so I can't simply say $\frac{Q_{Tot}}{4\pi\epsilon_0 r}$.

There are different ways of solving this.
My favorite method uses insight from summation of orthogonal fns.

For the B.C. we will use the z-axis at large values of z.

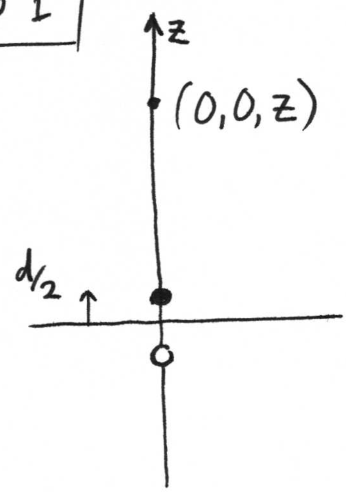
Overview of the strategy:

a) Calculate Φ on z-axis in limit of large r.

b) write $\Phi(r,\theta)$ as a sum of orthogonal fns.

c) Find unknown coefficients by matching the summation to the B.C.

Step 1



$$\Phi(0,0,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{z - d/2} - \frac{q}{z + d/2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{z}\left(\frac{1}{1 - d/2z} - \frac{1}{1 + d/2z}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{z}\left(1 + \frac{d}{2z} + \dots - \left(1 - \frac{d}{2z} + \dots\right)\right)$$

$$= \frac{qd}{4\pi\epsilon_0}\,\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)$$

Step 2

$$\Phi(r,\theta) = \sum_{l}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

$A_l = 0$ because $\Phi \to 0$ as $r \to \infty$

Along the positive z axis, the summation reduces to

$$\Phi(r, \theta = 0) = \frac{B_0}{z} P_0(1) + \frac{B_1}{z^2} P_1(1) + \frac{B_2}{z^3} P_2(1) + \dots$$

$\boxed{\text{Step 3}}$ Match the summation of orthogonal functions to the B.C.

$$\Phi(0,0,z) = \frac{qd}{4\pi\epsilon_0 z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)$$

$$\Phi(0,0,z) = \frac{B_0}{z} P_0(1) + \frac{B_1}{z^2} P_1(1) + \mathcal{O}\left(\frac{1}{z^3}\right)$$

$$\Rightarrow \quad B_0 = 0 \quad \text{and} \quad B_1 = \frac{qd}{4\pi\epsilon_0}$$

$\boxed{\text{Final answer}}$

$$\Phi(r,\theta) = \frac{qd}{4\pi\epsilon_0} \frac{\cos\theta}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

Summary: We noticed that potential drops as $\frac{1}{r^2}$ along the z axis at larger r.

We got θ-dependence "for free" because only one term in the summation of orthogonal functions matches the $1/r^2$ dependence.

Comment: There is a beautiful connection between r-dependence and θ-dependence. This connection is enforced by the condition $\nabla^2 \Phi = 0$.