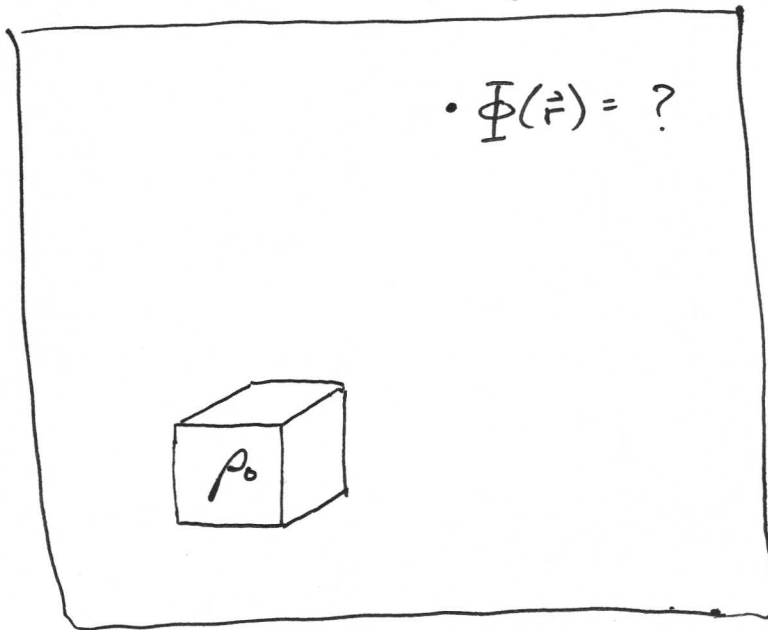


Pop Quiz

Day 15

2015



A cube with edge length a is filled with uniform charge density ρ_0 . ~~Give an~~ Write an ^{approximate} analytical solution for $\Phi(\vec{r})$ that approaches the true value of $\Phi(\vec{r})$ at large r .

DAY 15

PH 631

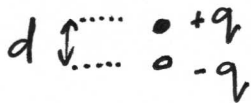
2015

Instructor
Ethan Minot

INTRODUCTION TO MULTIPOLE EXPANSIONS

Consider a similar question as the pop quiz:

$$\Phi(\vec{r}) = ?$$



Find an analytical expression, $\Phi_{\text{approx}}(r, \theta)$ that approaches the true value of $\Phi(\vec{r})$ at large \vec{r} .

Note: $Q_{\text{Tot}} = 0$, so I can't simply say $\frac{Q_{\text{Tot}}}{4\pi\epsilon_0 r}$.

There are different ways of solving this.

My favorite method uses insight from summation of orthogonal fns.

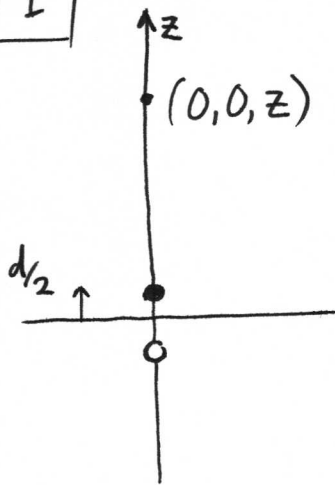
For the B.C. we will use the z -axis at large values of z .

(2)

Overview of the strategy:

- Calculate Φ on z -axis in limit of large r .
- Write $\Phi(r, \theta)$ as a sum of orthogonal fns.
- Find unknown coefficients by matching the summation to the B.C.

Step 1



$$\begin{aligned}
 \Phi(0, 0, z) &= \frac{1}{4\pi\epsilon_0} \left(\frac{q}{z - d/2} - \frac{q}{z + d/2} \right) \\
 &= \frac{q}{4\pi\epsilon_0} \frac{1}{z} \left(\frac{1}{1 - d/2z} - \frac{1}{1 + d/2z} \right) \\
 &= \frac{q}{4\pi\epsilon_0} \frac{1}{z} \left(1 + \frac{d}{2z} + \dots - \left(1 - \frac{d}{2z} + \dots \right) \right) \\
 &= \frac{q d}{4\pi\epsilon_0} \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)
 \end{aligned}$$

Step 2

$$\Phi(r, \theta) = \sum_l (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\theta)$$

$$A_l = 0 \quad \text{because } \Phi \rightarrow 0 \text{ as } r \rightarrow \infty$$

Along the positive z axis, the summation reduces to

$$\Phi(r, \theta=0) = \frac{B_0}{z} P_0(1) + \frac{B_1}{z^2} P_1(1) + \frac{B_2}{z^3} P_2(1) + \dots$$

(3)

Step 3 | match the summation of orthogonal functions to the B.C.

$$\Phi(0,0,z) = \frac{qd}{4\pi\epsilon_0 z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)$$

$$\Phi(0,0,z) = \frac{B_0}{z} P_0(1) + \frac{B_1}{z^2} P_1(1) + \mathcal{O}\left(\frac{1}{z^3}\right)$$

$$\Rightarrow B_0 = 0 \quad \text{and} \quad B_1 = \frac{qd}{4\pi\epsilon_0}$$

Final answer

$$\Phi(r,\theta) = \frac{qd}{4\pi\epsilon_0} \frac{\cos\theta}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

Summary: We noticed that potential drops as $\frac{1}{r^2}$ along the z axis at larger r .

We got θ -dependence "for free" because only one term in the summation of orthogonal functions matches the $\frac{1}{r^2}$ dependence.

Comment: There is a beautiful connection between r -dependence and θ -dependence. This connection is enforced by the condition $\nabla^2\Phi = 0$.