DAY 14

PH 631
2015

Instructor
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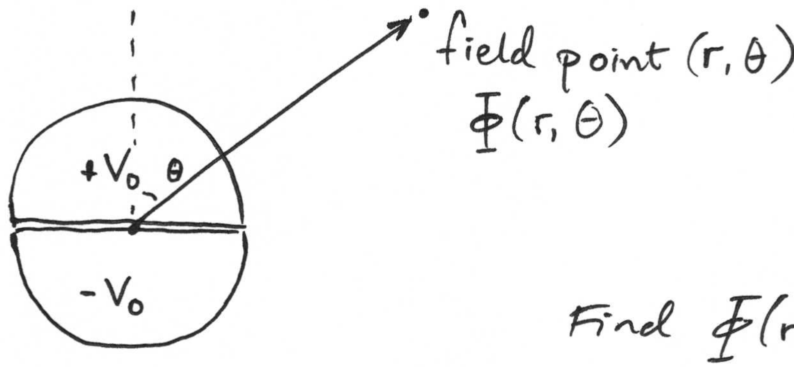
A metal sphere (radius "a") is cut into two hemispheres.



Azimuthal symmetry

Find $\Phi(r, \theta)$ for $r > a$

Step 1: Recognize which coord system to use. (make it easy to match B.C.s)

Then write $\Phi(r, \theta)$ as a sum of seperable fns

$$\Phi(r, \theta) = \sum_{\ell} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

Note, each term in the series satisfies $\nabla^2 f = 0$.

Step 2: If possible, reduce the number of terms in the summation.

In this case $A_\ell = 0$ for all ℓ.

<u>Step 3</u> : Find the value of non-zero unknown coefficients in the summation.

Note: We'll need the orthogonality relationship

$$\int_{-1}^{1} P_{\ell'}(x) P_\ell(x)\, dx = \frac{2}{2\ell+1}\, \delta_{\ell\ell'}$$

The B.C.s tell us $\Phi(r=a,\theta) = \begin{cases} +V_0 & 0<\theta<\pi/2 \\ -V_0 & \pi/2<\theta<\pi \end{cases}$

change variables to $x = \cos\theta$

$$\Phi(a, x) = \begin{cases} +V_0 & 0<x<1 \\ -V_0 & -1<x<0 \end{cases} \qquad \text{— ①}$$

The sum of orthogonal/seperable fns tells us

$$\Phi(r=a,x) = \sum_\ell B_\ell\, a^{-(\ell+1)} P_\ell(x) \qquad \text{— ②}$$

We will find B_ℓ by equating Eq ① and Eq ②.
You will often come across problems like this in graduate physics classes. The next two steps are always the same (please memorize)

- Multiply both sides of eq^n 2 by $P_{\ell'}(x)$
  i.e. one of the orthogonal functions
- Integrate both sides from $x=-1$ to $+1$.
  i.e. the range of the independent variable.

$$\int_{-1}^{1} P_{\ell'}(x)\,\Phi(a,x)\,dx = \int_{-1}^{1} P_{\ell'}(x) \sum_{\ell} B_\ell\, a^{-(\ell+1)} P_\ell(x)\,dx$$

use the orthogonality relationship.

$$= \sum_{\ell} B_\ell\, a^{-(\ell+1)} \frac{2}{2\ell+1}\,\delta_{\ell\ell'}$$

$$= B_{\ell'}\, a^{-(\ell'+1)} \frac{2}{2\ell'+1}$$

$$\Rightarrow B_\ell = \frac{a^{\ell+1}(2\ell+1)}{2} \int_{-1}^{1} P_\ell(x)\,\Phi(a,x)\,dx$$

Evaluate this expression for $\ell = 0, 1, 2, 3, 4$

$\ell = 0$ $\quad P_0(x) = 1$ which is an even function of x

$\Phi(a,x)$ is an odd function of x

$$\Rightarrow \int_{-1}^{1} P_\ell(x)\,\Phi(a,x)\,dx = 0$$

$$B_0 = 0$$

$l = 1$ $\quad P_1(x) = x$

$$B_1 = \frac{a^3\,3}{2} \int_{-1}^{1} \overbrace{x\,\Phi(a,x)}^{\text{even function of } x}\,dx$$

$$= \frac{3a^3}{2}\; 2\int_0^1 x V_0\,dx$$

$$= \frac{3a^2 V_0}{2}$$

$l = 2$ $\quad B_2 = 0$

$l = 3$

$$B_3 = \frac{a^4\,7}{2}\; 2\int_0^1 \frac{1}{2}(5x^3 - 3x)V_0\,dx$$

$$= -\frac{7}{8} V_0 a^4$$

$l = 4$ $\quad B_4 = 0$

Final answer

$$\Phi(r,\theta) = \frac{3V_0 a^2}{2}\frac{1}{r^2}P_1(\cos\theta) - \frac{7V_0 a^4}{8}\frac{1}{r^4}P_3(\cos\theta) + \cdots$$

Note that higher order terms, for example $B_6 r^{-6}$, will be very small corrections when $r \gg a$.