

PH 631

①  
Day 13  
2015

Instructor  
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summary / recap of what we've done regarding  
the "summation of orthogonal fns technique".

If solving  $\nabla^2 \Phi(x, y) = 0$   
↑ cartesian  
coords

use the family of fns  
 $\sin \alpha x e^{-\alpha y}$

If solving  $\nabla^2 \Phi(r, \phi) = 0$   
↑ polar  
coords

use the family of fns  
 $r^\alpha \sin \alpha \phi$

If solving  $\nabla^2 f(r, \theta) = 0$   
↑ spherical  
coordinates,  
no  $\phi$  dependence

use the family of fns  
 $r^l P_l(\cos \theta)$   
 $r^{-(l+1)} P_l(\cos \theta)$   
for  $l = 0, 1, 2, \dots$

where  $P_0(x) = 1$      $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$      $P_3(x) = \frac{1}{2}(5x^3 - 3x)$     ...

(2)

Notice that we choose functions that are separable in the given coordinate system.

For example:  $\sin \alpha x e^{-\alpha y} = \underbrace{f(x)g(y)}_{\text{separable function}}$

$$r^\alpha \sin \alpha \phi = \underbrace{u(r)v(\phi)}_{\text{separable function}}$$

This is what makes our choice unique.

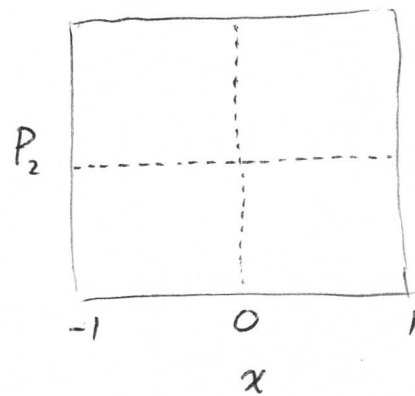
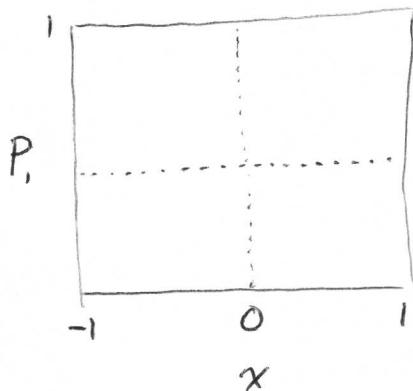
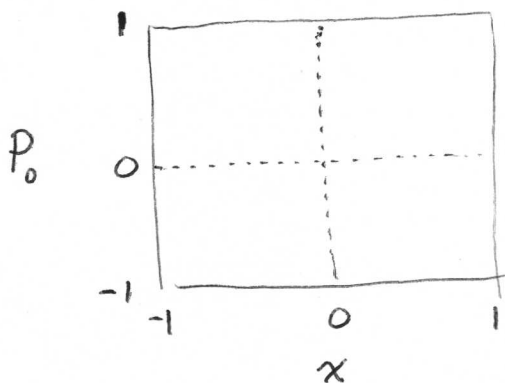
$\sin \alpha x e^{-\alpha y}$  etc. are the only functions that solve  $\nabla^2 \Phi(x, y) = 0$  and are separable into functions of  $x$  and  $y$ .

(3)

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Pop Quiz

a) Plot  $P_0(x)$ ,  $P_1(x)$  &  $P_2(x)$  on the range  $-1 \leq x \leq 1$



b) Show that

$$\int_{-1}^1 P_0(x) P_1(x) dx = 0$$

$$\int_{-1}^1 P_0(x) P_2(x) dx = 0$$

$$\int_{-1}^1 P_1(x) P_2(x) dx = 0$$

c) Show that  $\int_{-1}^1 |P_l(x)|^2 dx = \frac{2}{2l+1}$  for  $l=0, 1, 2$

(4)

Summarizing results from the pop quiz

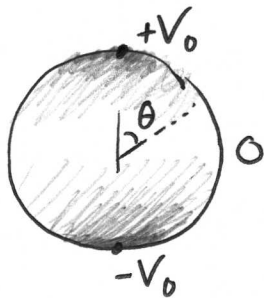
$$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'}$$

↑  
Kronecker delta,  
either zero or one.

This is the orthogonality relation for Legendre polynomials.

We'll use this relation in examples below.

EXAMPLE 1:



Spherical shell of radius "a" carries a charge density  $\sigma(\theta)$  (azimuthally symmetric).

The charge density is such that

$$\Phi(a, \theta) = V_0 \cos \theta$$

Note that  $\Phi \rightarrow 0$  as  $r \rightarrow \infty$

Find  $\Phi(r, \theta)$  outside the shell.