PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

The unknown fn is a fn of 2 or more variables.

You've done the hard work; you know how to solve ODEs.

Two new "tricks" will allow you to tackle PDEs.

① Switch to the appropriate coord system

If the unknown function is $f(x,y,z)$ you are free to recast the problem in terms of $f(r,\theta,\phi)$ etc.

② The appropriate coord system will (hopefully) facilitate "SEPARATION OF VARIABLES"

1 PDE goes to several ODEs.

The seperation of variable technique is covered in Butkov §8.1 – 8.4. Examples:
- Travelling waves on a stretched string
- Diffusion Eqn.

For class time we'll use another famous example, the hydrogen atom wavefunctions.

PDE: $$i\hbar \frac{\partial \Psi(x,y,z,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi(x,y,z,t) - \frac{e^2}{4\pi\epsilon_0 r}\Psi(x,y,z,t)$$

We can pick this apart, 1 variable at a time, reducing the problem to 4 ODEs.

The approach is very general, you will be using these steps in your hw.

$\boxed{\text{STEP 1}}$ Since $V = \frac{-e^2}{4\pi\epsilon_0 r}$ choose spherical coordinates, hopefully the final soln will have the form

$$\Psi(r,\theta,\phi,t) = f(t)R(r)\Theta(\theta)\Phi(\phi)$$

ANSATZ: Unknown fn is a product of 4 single-variable functions.

$\boxed{\text{STEP 2}}$ Plug the ansatz into the PDE, then isolate one variable on the left hand side (LHS

$$i\hbar \frac{\partial}{\partial t} f R \Theta \Phi = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] f R \Theta \Phi$$

$$i\hbar R \Theta \Phi \frac{\partial f}{\partial t} = f \left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] R \Theta \Phi$$

$$i\hbar \frac{1}{f} \frac{\partial f}{\partial t} = \frac{1}{R \Theta \Phi} \left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] R \Theta \Phi$$

STEP 3 Introduce the separation constant

Since the lefthand side (LHS) depends on time, and the RHS depends on space, equality at at times & locations is only possible if

LHS = const = RHS.

Call the separation const E (conventional name)

$$\boxed{i\hbar \frac{1}{f} \frac{df}{dt} = E} \qquad \text{ODE \#1}$$

$\boxed{\text{STEP 4}}$ Now we have a new PDE

$$RHS = E \qquad \text{——— PDE \#2}$$

Tackle this new PDE in the same fashion (try to isolate one variable).

Note that ∇^2 contains all three variables

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Let's try to isolate the r terms on the ~~R~~LHS of PDE #2.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\hbar^2}\Big[V(r)-E\Big]=-\frac{1}{\Theta}\,\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)$$

$$+\frac{1}{\Phi}\,\frac{1}{\sin^2\theta}\,\frac{d^2\Phi}{d\phi^2}$$

Now, repeat $\boxed{\text{STEP 3}}$ Introduce another separation constant.

This time it is conventional to call the const $\ell(\ell+1)$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\hbar^2}\Big[V(r)-E\Big]=\ell(\ell+1) \quad \text{——— ODE \#2}$$

Now, repeat $\boxed{\text{STEP 4}}$ with PDE #3

$$\hookrightarrow \ell(\ell+1)+\frac{1}{\Theta}\sin\theta\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)=\frac{1}{\Phi}\,\frac{d^2\Phi}{d\phi^2}$$

Now repeat $\boxed{\text{STEP 3}}$ Introduce another seperation const.

$$\ell(\ell+1)+\frac{1}{\Theta}\sin\theta\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)=m^2 \qquad \text{——— ODE \#3}$$

$$\frac{1}{\Phi}\,\frac{d^2\Phi}{d\phi^2}=m^2 \qquad \text{——— ODE \#4}$$

We've reduced the original PDE to a set of 4 ODEs.

Solutions to the 4 ODEs are connected to one another via the separation constants

$$E, \ \ell(\ell+1) \text{ and } m^2.$$