POWER SERIES TECHNIQUE TO SOLVE O.D.E.s (continued)

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \qquad \text{[Legendre Eq}^n\text{]}$$

Example 3 of §3.5 Butkov.

For physical situations, λ is usually $\ell(\ell+1)$ where $\ell = 0, 1, 2 \ldots$ e.g. H atom wavefns.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$\Rightarrow (1-x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} c_n n x^{n-1} + \lambda \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} c_n n(n-1)x^n - \sum_{n=0}^{\infty} 2c_n n x^n + \sum_{n=0}^{\infty} \lambda c_n x^n = 0$$

$$c_{n+2}(n+2)(n+1) - c_n n(n-1) - 2c_n n + \lambda c_n = 0$$

$$c_{n+2}(n+2)(n+1) = c_n\left[n(n-1) + 2n - \lambda\right]$$

$$\boxed{c_{n+2} = c_n \frac{n(n-1) + 2n - \lambda}{(n+2)(n+1)}}$$

The recursion relation applies between odd terms, or between even terms, the coefficients don't mix together. This makes the analysis easier.
c_0 and c_1 can be the arbitrary consts in our general sol^n.

Consider the sol^{n}s for some different values of $\lambda = \ell(\ell+1)$

If $\ell = 0$, such that $\lambda = 0$

Either c_0, $c_1 = 0$

$c_0 \downarrow c_2 = 0 \downarrow c_4 = 0 \ \vdots$

$c_1 = 0 \downarrow c_3 = 0 \downarrow$

$\boxed{y(x) = c_0}$

OR $c_0 = 0$, c_1
$\downarrow$
messy, ill-behaved power series.

If $\ell = 1$, such that $\lambda = 2$

Either $c_0 = 1$, $c_1 = 0$
$\downarrow$
messy

OR $c_0 = 0$, c_1

$c_0 = 0 \downarrow c_2 = 0 \downarrow \vdots$

$c_1 \downarrow c_3 = 0 \downarrow \vdots$

$\boxed{y(x) = c_1 x}$

If $l = 2$, such that $\lambda = 6$

$\boxed{\text{Either}}$ c_0 $\qquad c_1 = 0$

$c_2 = -3c_0 \qquad c_3 = 0$

$c_4 = 0$

$$\boxed{y(x) = c_0(1 - 3x^2)}$$

$\boxed{\text{OR}}$ $c_0 = 0 \qquad c_1$

$c_2 = 0 \qquad c_3 = \ldots$

messy

If $l = 3$

$$\boxed{y(x) = c_1\left(x - \frac{10}{6}x^3\right)}$$

If $l = 4$

$$\boxed{y(x) = c_0\left(1 - 10x^2 + \frac{35}{3}x^4\right)}$$

these functions are called the Legendre polynomials.

One more point about power series solutions before you know the crucial tricks/pitfalls.

Consider the Bessel Eqⁿ which arises when studying cylindrical waveguides (among other things).

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

In physical situations ν is integer or $\frac{1}{2}$ integer.

Write in standard form

$$y'' + \underbrace{\frac{y'}{x} + \frac{(x^2 - \nu^2)}{x^2} y}_{} = 0$$

These terms diverge at $z = 0$.
The D.E. has a singularity at $z = 0$.

The power series idea can still work if we assume

$$y(x) = \sum_{n=0}^{\infty} x^{s} c_n x^n$$

and find the appropriate value of s. "The Frobenius Method"

To illustrate the method, we will take $\nu = 1/2$ in the Bessel eqn.

[Example 2, §3.5 Butkov]

$$x^2 y'' + x y' + \left(x^2 - \tfrac{1}{4}\right) y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^{s+n}$$

$$y' = \sum_{n=0}^{\infty} c_n (s+n) x^{s+n-1}$$

Summation still starts at zero. If $s+n=0$ such a term will naturally drop out.

$$y'' = \sum_{n=0}^{\infty} c_n (s+n)(s+n-1) x^{s+n-2}$$

$$\sum_{n=0}^{\infty} c_n \left[(s+n)(s+n-1) + (s+n) - \tfrac{1}{4}\right] x^{s+n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{s+n+2}}_{= \sum_{n=2}^{\infty} c_{n-2} x^{s+n}} = 0$$

$$c_0 \left[s(s-1) + s - \tfrac{1}{4}\right] x^s + c_1 \left[(s+1)(s) + (s+1) - \tfrac{1}{4}\right] x^{s+1}$$

$$+ \sum_{n=2}^{\infty} \left(c_n \left[(s+n)(s+n-1) + (s+n) - \tfrac{1}{4}\right] + c_{n-2} \right) x^{s+n} = 0$$

The coefficient multiply each power of x must be zero.

Also we are going to pick s such that $c_0 \neq 0$.

Consider x^s term:

$$c_0\left[s(s-1)+s-\tfrac{1}{4}\right]=0$$

$$s^2-\tfrac{1}{4}=0 \qquad \text{"indicial eq}^{n}\text{"}$$

$$s=\pm\tfrac{1}{2}$$

Consider x^{s+1} term:

$$c_1\left[(s+1)s+(s+1)-\tfrac{1}{4}\right]=0$$

$$c_1\left[(s+1)^2-\tfrac{1}{4}\right]=0$$

If $s=\tfrac{1}{2}$ then $c_1=0$

If $s=-\tfrac{1}{2}$ then c'_1 is unconstrained.

The other terms give a recursion relation between c_n and c_{n-2}

If $s=\tfrac{1}{2}$ $\qquad c_n=-\dfrac{1}{n(n+1)}c_{n-2} \qquad n\geq 2$

If $s=-\tfrac{1}{2}$ $\qquad c'_n=-\dfrac{1}{n(n-1)}c'_{n-2} \qquad n\geq 2$

We can use $s = -1/2$ to find the general soln of this D.E.

c_0' & c_1' will be our 2 arbitrary constants.

$$y(x) = c_0' x^{-1/2}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$+ c_1' x^{-1/2}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= c_0' x^{-1/2} \cos x + c_1' x^{-1/2} \sin x$$

These functions are Bessel fns of order $1/2$ and $-1/2$.