PH561

DAY 5

HOMEWORK #2 HINTS

ON YOUR BOARDS

Find the phase difference between

$$\frac{2}{1+i}e^{-i\omega t} \quad \text{and} \quad \frac{8}{1-i}e^{-i\omega t}$$

RECIPE FOR APPLYING THE GREEN'S FN TECHNIQUE

① Identify the inhomogeneous D.E. that your unknown function must satisfy

$$\mathcal{L}\, u(x) = f(x)$$

Linear differential operator

Make note of any boundary condition that $u(x)$ must satisfy.

② Find the Green's function that is unique to the particular $\mathcal{L}$ and the particular boundary conditions in your problem.

$$\mathcal{L}\, G(x,\xi) = \delta(x-\xi)$$

where $G(x,y)$ satisfies the same B.C.s as the unknown fn $u(x)$.

Note that the RHS = 0 when $x<\xi$ & $x>\xi$ so the Green's fn will likely splice together solns of the homogeneous eqn.

③ Find $u(x)$ by calculating the integral

$$u(x) = \int_a^b \underbrace{f(\xi)\,d\xi}_{\substack{\text{Called an impulse} \\ \text{if it has dimensions Force}\cdot\text{Time}}}\; G(x,\xi)$$

Note: Integrate over ξ to leave a fn of x.

GREEN'S FUNCTION TECHNIQUE (CONTINUED)

Last time:

Apply Green's function technique to solve

$$\frac{d^2u}{dx^2} = \frac{F(x)}{T} \quad \text{where} \quad u=0 \text{ at } x=0, x=L.$$

$$(\mathcal{L}u = f(x))$$

STEP 1: Find Green's fn, i.e. $G(x,\xi)$ s.t.

$$\mathcal{L}G = \frac{d^2G}{dx^2} = \delta(x-\xi)$$

we got to the point

$$G = \begin{cases} \frac{A}{\xi}x & x<\xi \\ \frac{A(L-x)}{(L-\xi)} & x>\xi \end{cases}$$

i.e. solving the homogeneous eqn to the left and right of $x=\xi$.

to complete step 1, we need to find A.

set A such that

$$\frac{d^2G}{dx^2} = \delta(x-\xi)$$

We do this by integrating over the δ-fn

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \frac{d^2G}{dx^2} = \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)$$

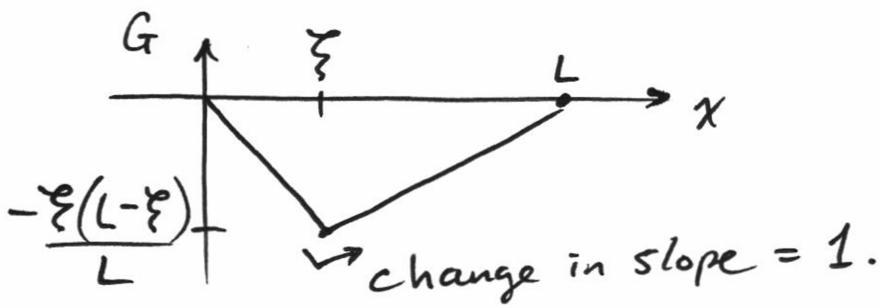
$$\left.\frac{dG}{dx}\right|_{x=\xi+\epsilon} - \left.\frac{dG}{dx}\right|_{x=\xi-\epsilon} = 1$$

$$\left(\frac{-A}{L-\xi}\right) - \frac{A}{\xi} = 1$$

$$\Rightarrow \quad A = \frac{-\xi(L-\xi)}{L}$$

Plug this back into the $G(x,\xi)$ eqn to find

$$G(x,\xi) = \begin{cases} -\dfrac{x(L-\xi)}{L} & , \; x<\xi \\[2ex] -\dfrac{\xi(L-x)}{L} & , \; x>\xi \end{cases}$$



Now we know the Green's fn, we can use it to find the curve of the string under any load ~~pattern~~ distribution (i.e. changing the source term in the DE).

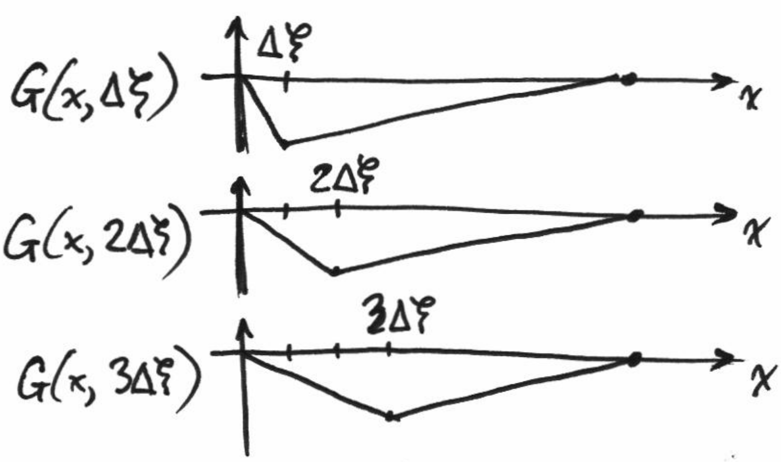
Example

$$\frac{d^2u}{dx^2} = \frac{F_0}{T} \quad \text{(same force at every position).}$$

Solution using a computer.

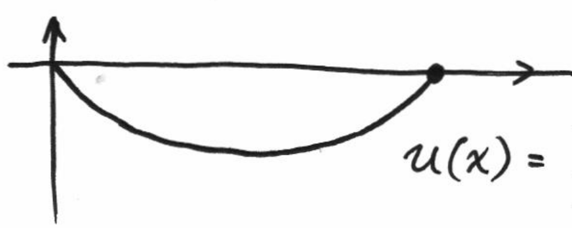
$\boxed{1.}$ Choose a small $\Delta\xi$

$\boxed{2.}$ Evaluate $G(x, \xi)$ for $\xi = 0, \Delta\xi, 2\Delta\xi, \ldots L$



$\vdots$

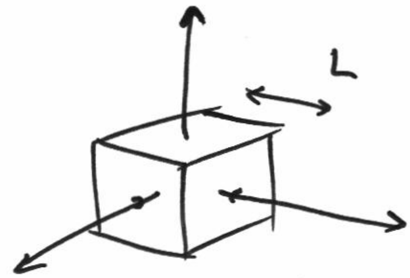
$\boxed{3.}$



$$u(x) = \sum_{i=1}^{N} G(x, \xi_i) F_0 \Delta\xi$$

GREEN'S FUNCTION FOR SOLVING PDEs

Example: $\nabla^2 \phi = \dfrac{\rho_{cube}(x,y,z)}{\epsilon_0}$



ρ_0 inside
0 outside

B.C. $\phi = 0$ at infinity.

Solution: (Follow the recipe)

Step 1: $\mathcal{L} = \nabla^2$

Step 2: $\nabla^2 G(\vec{r}, \vec{\xi}) = \delta(\vec{r} - \vec{\xi})$, find G.

Try $G(\vec{r}, \vec{\xi}) = \dfrac{1}{|\vec{r} - \vec{\xi}|}$ (it is consistent with the B.C.)

Does $\nabla^2 G = 0$ when $\vec{r} \neq \vec{\xi}$?

Please check this explicitly using Cartesian coordinates.