In HW #1 you practiced manipulating complex numbers. Now tie this into Diff. Eqns.

§2.5 Butkov "Applications of Euler's Formula"

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

Suggests that we can describe sinusoidally varying physical ~~reals~~ quantities using the real component of $re^{i\theta}$:

$$a\cos(\omega t - \theta) = \mathrm{Re}\left\{\underbrace{a e^{i\theta} e^{-i\omega t}}\right\}$$

complex number that is a function of time.

Often used when solving diff. eqns. where all variables are real numbers.

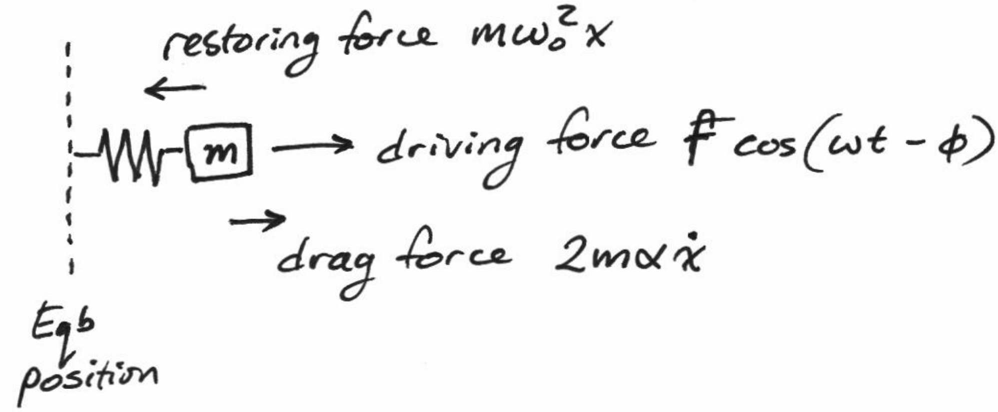
Example:

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = F\cos(\omega t - \phi) \qquad (1)$$

where all variables are real numbers

Classified as ______ ______ ______ diff. eqn.

Interpret Eq. 1 using Newton's 2^{nd} Law:



Eq. 1 is called the damped driven harmonic oscillator eq^n.

- The brute force solution method is

  a) Guess the sol^n: $x(t) = A\cos(\omega t - \varphi)$

  A will depend on F & ω; φ phase-shifted from ϕ.

  b) Plug into 1 and spend the rest of the week using trig identities to determine A & φ.

- The more elegant alternative is

  Let $x(t)$ be complex, $x(t) = \mathrm{Re}\, x(t) + i\, \mathrm{Im}\, x(t)$

  Note that $\dfrac{dx}{dt} = \dfrac{d}{dt}\mathrm{Re}\, x(t) + i\dfrac{d}{dt}\mathrm{Im}\, x(t)$

  $$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}\mathrm{Re}\, x(t) + i\frac{d^2}{dt^2}\mathrm{Im}\, x(t)$$

3

This means Eq. 1 (the damped driven HO eqⁿ) can be expressed as

$$\mathrm{Re}\left[\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x\right] = \mathrm{Re}\left[F e^{i\phi} e^{-i\omega t}\right]$$

where $x(t)$ is complex, we solve for $x(t)$, but in the end we'll only be interested in $\mathrm{Re}\, x(t)$.

So, if we can solve

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = \tilde{F} e^{-i\omega t} \qquad (2)$$

we will be done.

As before,
We guess that $x(t) = A e^{i\psi} e^{-i\omega t} = \tilde{A} e^{-i\omega t}$

Plug this guess into Eq. 2.

$\boxed{\text{ON YOUR BOARDS}}$

3 groups try the hard way.
3 groups try the elegant way.

Final answer

$$\mathrm{Re}\left\{ x(t) \right\} = \mathrm{Re}\left\{ \frac{F e^{-i(\omega t - \phi)}}{(\omega_0^2 - \omega^2) - i 2\alpha\omega} \right\}$$

HW #2

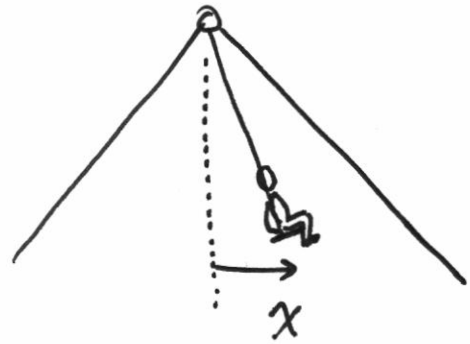
We will return to the damped-driven HO when covering Green's fns. ~~Later~~ You will want to know the amplitude of $\mathrm{Re}\{x(t)\}$ when the driving force is on resonance, $F\cos(\omega_0 t - \phi)$

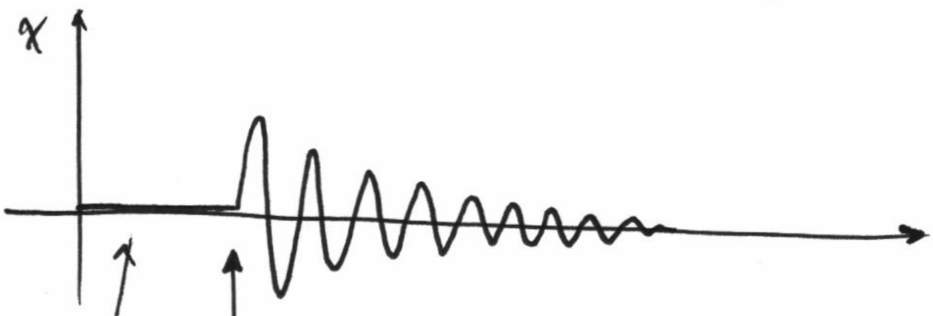
natural resonance of the system.

INTRO TO GREEN'S FUNCTIONS ⑤ §12.1 Butkov

"Cartoon" illustration of the Green's fn technique

Pushing a kid on a swing





Sitting at equilibrium is one soln to the D.E. when no one is pushing.

One sudden push at $t = 10s$. Force in a δ-fn. Resulting change in momentum is Δp.

$$x_1(t) = \begin{cases} 0 & t < 10s \\ \frac{\Delta p}{m\omega} e^{-\frac{t-10s}{t_0}} \sin(\omega_0(t-10s)) & \end{cases}$$

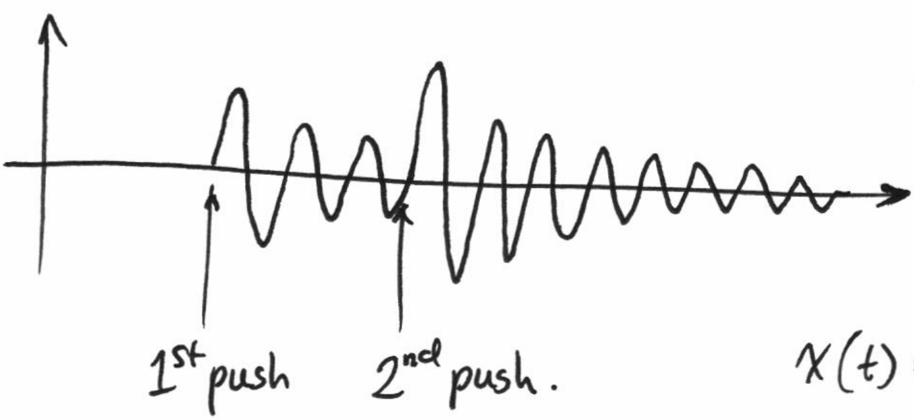
One sudden push at $t = 20s$. Same force.

$$x_2(t) = \begin{cases} 0 \\ \frac{\Delta p}{m\omega} e^{-\frac{t-20s}{t_0}} \sin(\omega_0(t-20 \end{cases}$$

ON YOUR BOARDS

During the time periods with no pushing, is the D.E. governing $x(t)$ a homogeneous eq^n (can solutions be superimposed)?

What happens to a kid who receives two pushes?



$$x(t) = x_1(t) + x_2(t)$$

we are adding together two Green's fns:

$$x(t) = \Delta p \, G(t, 10s) + \Delta p \, G(t, 20s)$$

where

$$G(t, \tau) = \begin{cases} 0 & t < \tau \\ \frac{1}{m\omega} \exp\left(\frac{t-\tau}{t_0}\right) \sin\left(\omega_0 (t-\tau)\right) & t > \tau \end{cases}$$