PH 561

Eigenfunction method for solving linear differential equations

$$\boxed{\mathcal{L}\, y(x) = f(x)} + \text{B.C.}$$

a linear differential operator acting on $y(x)$

some general function

In general it is not possible to solve this problem unless $f(x)$ is known and simple, even if we can solve homogeneous version of this equation.

The way to solve this problem is based on:
- linearity of $\mathcal{L}$
- re-writing the general solution as a superposition of some set of functions $\{y_i(x)\}$ that each individually satisfy the B.C. and

$$\underbrace{\mathcal{L}\, y_i(x) = \lambda_i\, y(x)}$$

looks like $A\vec{x} = \lambda \vec{x}$

Ex. $$-\frac{d^2y}{dx^2} = \omega^2 y$$

$$\mathcal{L} = -\frac{d^2y}{dx^2} \qquad \lambda = \omega^2$$

Some linear algebra

<u>Vector space</u>, $\vec{u}, \vec{v} \in V$

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{0} + \vec{v} = \vec{v}$
- $\vec{v} + (-\vec{v}) = \vec{0}$
- $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- $(a+b)\vec{u} = a\vec{u} + b\vec{u}$
- $(ab)\vec{u} = a(b\vec{u})$
- $1 \cdot \vec{v} = \vec{v}$

Set of functions $\int_a^b |f(x)|^2 dx < +\infty$

<u>Inner product space</u> = vector space + inner product and norm induced by this product

- $\langle u | v \rangle = \langle v | u \rangle^*$
- $\langle au | v \rangle = a \langle u | v \rangle$
- $\langle u + v | w \rangle = \langle u | w \rangle + \langle v | w \rangle$
- $\langle u | u \rangle \geq 0$, $\langle u | u \rangle = 0$ iff $u = 0$

$$\langle f | g \rangle = \int_a^b f^*(x) g(x) dx$$

$$\| f \| = \sqrt{\langle f | f \rangle}$$

<u>Hilbert space</u> = inner product space that is <u>complete</u>

Basis

Vectors	Functions
• Any set of N linearly independent vectors form a basis $\{\vec{e}_i\}$, $N = \dim V$	
• Any vector can be re-writen in a given basis: $\vec{x} = \sum_{i=1}^{N} x_i \vec{e}_i$	

• Gram - Schmidt orthonormalization

If $y_n(x)$, $n = 1, 2, \ldots$, are linearly independent (but not orthonormal) basis for the Hilbert space, then an orthonormal set of basis functions may be produced:

$$\phi_0 = y_0 \qquad \hat{\phi}_0 = \frac{y_0}{\|y_0\|}$$

$$\phi_1 = y_1 - \hat{\phi}_0 \langle \hat{\phi}_0 | y_1 \rangle \qquad \hat{\phi}_1 = \frac{\phi_1}{\|\phi_1\|}$$

$$\phi_2 = y_2 - \hat{\phi}_1 \langle \hat{\phi}_1 | y_2 \rangle - \hat{\phi}_0 \langle \hat{\phi}_0 | y_2 \rangle \qquad \hat{\phi}_2 = \frac{\phi_2}{\|\phi_2\|}$$

$$\vdots$$

$$\boxed{\phi_n = y_n - \sum_{k=1}^{n-1} \hat{\phi}_k \langle \hat{\phi}_k | y_n \rangle}$$

<u>Problem</u>: Starting from linearly independent functions $y_n(x) = x^n$, $n = 0, 1, \ldots$ construct three orthonormal functions over the interval $[-1, 1]$.