HEAT TRANSPORT

First, review website to look at the techniques we've covered.

Note, we have not covered vector calculus or linear algebra. EM & QM are covering?

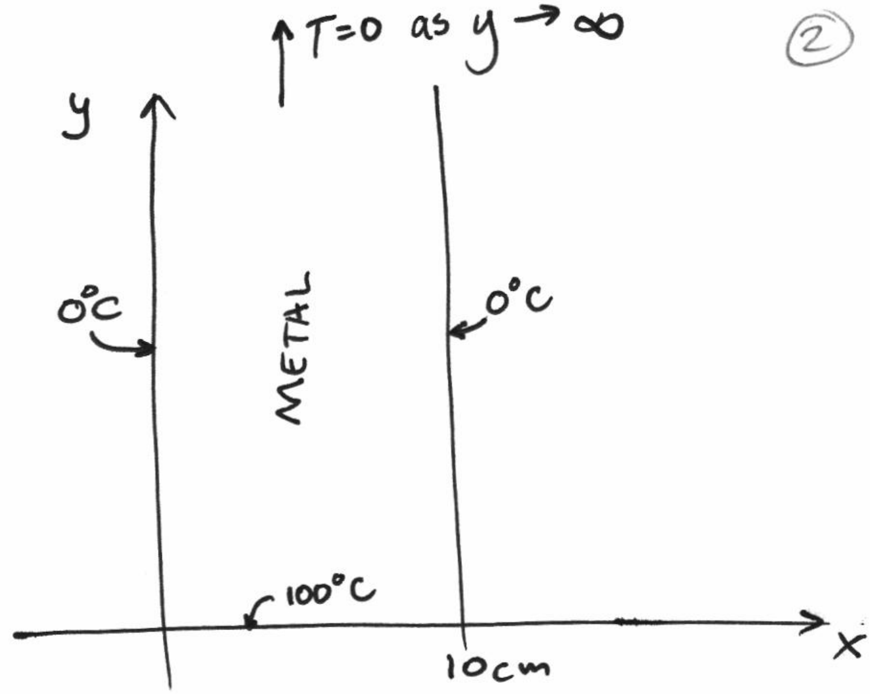
I want to select 4 or 5 physical situations that allow us to practice / fill in gaps / solidify what we've covered.

CASE STUDY 1

Temperature distribution over a strip of sheet metal.

[For another heat flow problem see Butkov §8.5 Example 2]

- Seperation of variables
- Solving ODEs
- Fourier series.



Find $T(x,y)$, temperature of the metal, assuming that system has reached steady state conditions. There is no heat flow in the z-direction.

First, classify the type of D.E. we must solve

Linear	Homogeneous	1st order	Partial
Non Linear	Inhomogeneous	2nd order	Ordinary
		⋮	

There is only one ~~way~~ method available to get started.

Take the first steps on your own.

We'll call the separation constant K^2.

What do we know about it so far? K^2 is a real number.

If $K^2 > 0$

$$\frac{d^2X}{dx^2} = K^2X \qquad \frac{d^2Y}{dy^2} = -K^2Y$$

$$\underbrace{X = Ae^{Kx} + Be^{-Kx}} \qquad \underbrace{Y = C\sin Ky + D\cos Ky}$$

check that I have the correct no. of arbitrary consts and linearly independent fns.

If $K^2 = 0$

If $K^2 < 0$

The B.C.s put important restrictions on the value of the separation const.

B.C. #1 $T=0$ as $y \to \infty$

$K^2 > 0, \quad Y = C \sin Ky + D \cos Ky$? No.

$K^2 = 0, \quad Y = Cy + D$? No.

$K^2 < 0, \quad Y = Ce^{Ky} + De^{-Ky}$? OK if $C = 0$.

$\Rightarrow K^2$ must be negative.

This means $X = A \sin Kx + B \cos Kx$.

B.C. #2 $T=0$ at $x=0$ Only possible if $B=0$.

$$X = A \sin Kx$$

B.C. #3 $T=0$ at $x = 10\text{cm}$ Only possible if

$$K = \frac{n\pi}{10\text{cm}} \qquad n = 1, 2, 3 \ldots$$

Recall that $T(x,y) = X(x)Y(y)$

Therefore we can make a list of functions that will satisfy $\nabla^2 T = 0$ and B.C. #1, 2, 3.

$$\left\{\begin{array}{l} D_1 e^{-\frac{\pi}{10}y} A_1 \sin\frac{\pi x}{10}, \\ D_2 e^{-\frac{2\pi}{10}y} A_2 \sin\frac{2\pi x}{10}, \\ D_3 e^{-\frac{3\pi}{10}y} A_3 \sin\frac{3\pi x}{10}, \\ \vdots \\ D_n e^{-\frac{n\pi}{10}y} A_n \sin\frac{n\pi x}{10} \end{array}\right\}$$

where x & y are in cm.

Set of functions that satisfy $\nabla^2 T = 0$ & B.C. 1,2,3.

Because $\nabla^2 T$ is a homogeneous D.E. we can create a ~~general~~ soln from any linear combination of functions in this set.

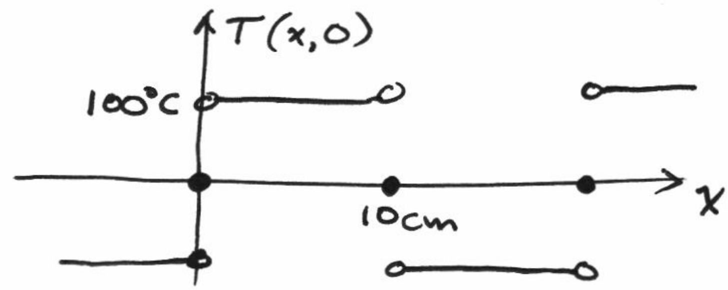
$$T(x,y) = \sum_{n=1}^{\infty} b_n e^{-\frac{n\pi}{10}y} \sin\left(\frac{n\pi x}{10}\right)$$

Now, there is one more B.C. The last B.C. will constrain the values of b_n.

B.C. #4 $T = 100°C$ at $y = 0$

$$T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) = 100°C \quad \text{for } 0 < x < 10\,\text{cm}$$

Choose b_n so that this sum converges to a square wave



Recall formula to find coeffs for a Fourier sine series.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

In this case $\quad b_n = \frac{2}{L}\int_0^L 100°C \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{400°C}{n\pi} \quad \text{where } n = 1, 3, 5 \ldots$$

$$\boxed{T(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{400°C}{n\pi} e^{-\frac{n\pi}{10}y} \sin\left(\frac{n\pi x}{10}\right)}$$

where x & y are in cm.

Sketch isotherms.

- Consider $T(x, @\ \text{large } y)$
- Consider corners.
- Consider midline $T(5, y)$.