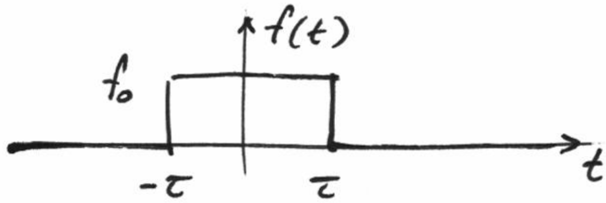
Last time: Find $x(t)$ when the force



is applied to a harmonic oscillator which is initially at rest.

$$x(t) = \frac{f_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega\tau \, e^{-i\omega t}}{\omega(\omega_0^2 - \omega^2)} \, d\omega$$

[This integral eqn was obtained by applying a F.T. to the Diff. Eqn.]

Note that τ is a constant. And t is a constant as far as the integration is concerned.

Using $\sin\omega\tau = \frac{1}{2i}\left(e^{i\omega\tau} - e^{-i\omega\tau}\right)$ we can write

$$x(t) = \frac{f_0}{2i\pi}\left[\int_{-\infty}^{\infty} \frac{e^{i\omega(\tau - t)}}{\omega(\omega_0^2 - \omega^2)} \, d\omega - \int_{-\infty}^{\infty} \frac{e^{-i\omega(\tau + t)}}{\omega(\omega_0^2 - \omega^2)} \, d\omega\right]$$

WARNING! By splitting the integrand we made pieces that have an additional pole at $\omega = 0$.

The boundedness of the integrand changes as we evaluate $x(t)$ for different ranges of t.

$\boxed{\text{CASE 1}}$ $t < -\tau$

This means $-t > \tau$

$$\tau - t > 2\tau$$

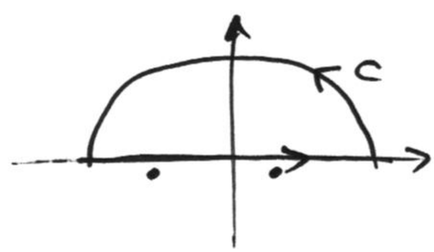
$e^{i\omega(\tau-t)}$ is bounded for ω above the real axis.

Similarly $t + \tau < 0$

$e^{-i\omega(t+\tau)}$ is bounded for ω above the real axis.

Therefore, we can close the contour "up".

$$x(t) = \frac{f_0}{\pi} \oint_C \frac{\sin \omega\tau \, e^{-i\omega t}}{\omega(\omega_0^2 - \omega^2)} \, d\omega$$



$= 0$ Because no residues inside the contour.

$\boxed{\text{CASE } 2}$ $\quad -\tau < t < \tau$

$$\tau > -t > -\tau$$

$$2\tau > \tau - t > 0$$

$e^{i\omega(\tau - t)}$ is bounded for ω above the the real axis.

$$0 < \tau + t < 2\tau$$

$e^{-i\omega(\tau + t)}$ is bounded for ω below the real axis.

Therefore, we need to use the split form of the integrand.

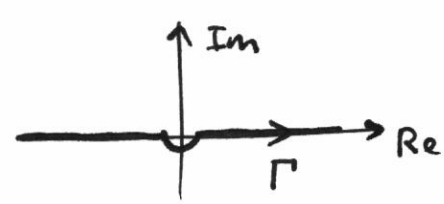
What do we do about the "new" poles at $\omega = 0$?

Note that

Equality holds because there is no singularity at $\omega = 0$.

$$\underset{(\text{real axis})}{\int_{-\infty}^{\infty}} \frac{\sin \omega\tau \, e^{-i\omega t}}{\omega(\omega_0^2 - \omega^2)} \, d\omega = \int_{\Gamma} \frac{\sin \omega\tau \, e^{-i\omega t}}{\omega(\omega_0^2 - \omega^2)} \, d\omega$$

where

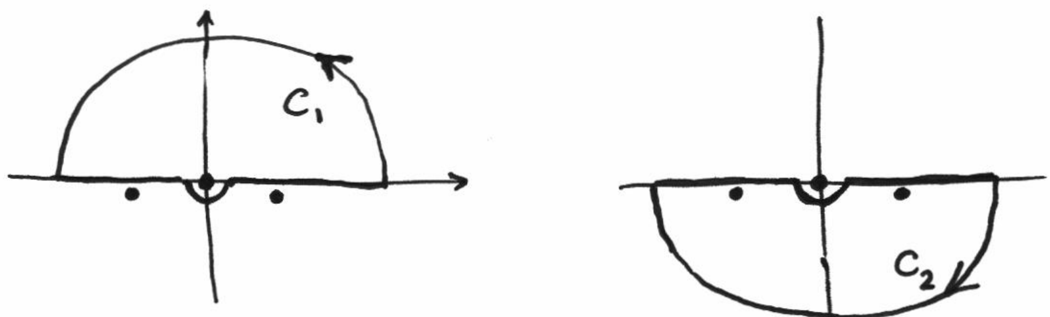


The path Γ avoids $\omega = 0$ by a small semicircle of radius $\epsilon \longrightarrow 0$

This means we can write

$$x(t) = \frac{f_0}{\pi} \int_{\Gamma} \frac{\sin\omega\tau \; e^{-i\omega t}}{\omega(\omega_0^2 - \omega^2)} \, d\omega$$

$$= \frac{f_0}{2i\pi} \left[\oint_{C_1} \frac{e^{i\omega(\tau - t)}}{\omega(\omega_0^2 - \omega^2)} \, d\omega - \oint_{C_2} \frac{e^{-i\omega(\tau + t)}}{\omega(\omega_0^2 - \omega^2)} \, d\omega \right]$$



(Counter clockwise)

(clockwise)

$$= \frac{f_0}{2i\pi} \left[2\pi i \; \text{Res} \; f_1(\omega = 0) + 2\pi i \; \text{Res} \; f_2(\omega = -\omega_0) + 2\pi i \; \text{Res} \; f_2(\omega = \omega_0) \right]$$

Evaluate these three residues.

$$\text{Res} \; f_1(0) = \lim_{\omega \to 0} \frac{\cancel{\omega} \, e^{i\omega(\tau - t)}}{\cancel{\omega}(\omega_0^2 - \omega^2)} = \frac{1}{\omega_0^2}$$

$$\text{Res} \; f_2(-\omega_0) = \lim_{\omega \to -\omega_0} \frac{\cancel{(\omega + \omega_0)} \, e^{-i\omega(\tau - t)}}{\omega \cancel{(\omega_0 + \omega)}(\omega_0 - \omega)} = \frac{e^{+i\omega_0(\tau + t)}}{-\omega_0(2\omega_0)}$$

$$\text{Res} \; f_2(\omega_0) = \lim_{\omega \to \omega_0} \frac{(\omega - \omega_0) \, e^{-i\omega(\tau - t)}}{\omega(\omega_0 + \omega)(\omega_0 - \omega)} = \frac{-e^{-i\omega_0(\tau + t)}}{\omega_0(2\omega_0)}$$

Adding together these three residues

$$x(t) = f_0\left[\frac{1}{\omega_0^2} - \frac{1}{2\omega_0^2}\left(e^{i\omega_0(\tau+t)} + e^{-i\omega_0(\tau+t)}\right)\right]$$

$$= \frac{f_0}{\omega_0^2}\left[1 - \cos\omega_0(\tau+t)\right] \qquad \text{for } -\tau<t<\tau$$

Is this consistent with what we expected?

Please compare to our preliminary analysis.

Conclusion: It was fun to practice these techniques, but completely unnecessary for this particular problem.