PH 561

DAY 14

FOURIER TRANSFORMS

First, quick review of Fourier series, see link on website

etc...

Last time, we "derived" Fourier cosine transform (heuristic derivation) from a cosine Fourier series.

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

series $\Rightarrow$ CONTINUUM

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(k)\cos kx\, dk$$

$$F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos kx\, dx$$

Today, derive the Fourier Transform from the complex form of a Fourier series.

COMPLEX FORM OF A FOURIER SERIES §4.5 Butkov.

starting point: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2}\exp\left(i\frac{n\pi x}{L}\right) + \frac{a_n}{2}\exp\left(-i\frac{n\pi x}{L}\right) + \frac{b_n}{2i}\exp\left(i\frac{n\pi x}{L}\right) - \frac{b_n}{2i}\exp\left(-i\frac{n\pi x}{L}\right)\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{\frac{a_n - ib_n}{2}\exp\left(i\frac{n\pi x}{L}\right) + \frac{a_n + ib_n}{2}\exp\left(-i\frac{n\pi x}{2}\right)\right\}$$

i.e. A sum of e^{ix} terms can make a sum of $\sin x$ & $\cos x$ terms.

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{a_n - ib_n}{2}\exp\left(i\frac{n\pi x}{L}\right) + \sum_{n=-1}^{-\infty}\frac{a_{-n} + ib_{-n}}{2}\exp\left(i\frac{n\pi x}{L}\right)$$

I flipped the sign of the index in the sum & then compensated in the n^{th} term expression.

$$= \sum_{n=-\infty}^{\infty} c_n \exp\left(i\frac{n\pi x}{L}\right)$$

Shorter, more elegant formula.

If $f(x)$ is even, $c_n = c_{-n}$

If $f(x)$ is odd, $c_n = -c_{-n}$

In general $f(x)$ doesn't have to be even or odd.

The Fourier Transform is the continuum limit of the complex form of the Fourier series.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ikx}, \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-ikx} dx$$

$$\left(\text{Remember } k = \frac{n\pi}{L}\right)$$

As $L \to \infty$

$$f(x) = \int_{-\infty}^{\infty} c_n e^{ikx} \frac{dk}{\pi/L}, \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} L c_n e^{ikx} dk, \quad \text{where} \quad L c_n = \frac{1}{2}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

I want symmetry between these prefactors.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int \frac{\sqrt{2}\, L c_n}{\sqrt{\pi}} e^{ikx} dk, \quad \text{where} \quad \frac{\sqrt{2}\, L c_n}{\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

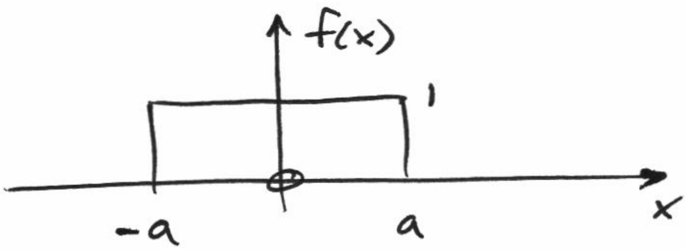
Finally, we can write in the standard form

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k)\, e^{-ikx}\, dk \\ F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{+ikx}\, dx \end{aligned}}$$

§7.1 Butkov

[following Butkov, I switched the sign of k to be consistent with modern convention].

Let's return to the top hat example



What is the Fourier transform representation of $f(x)$?