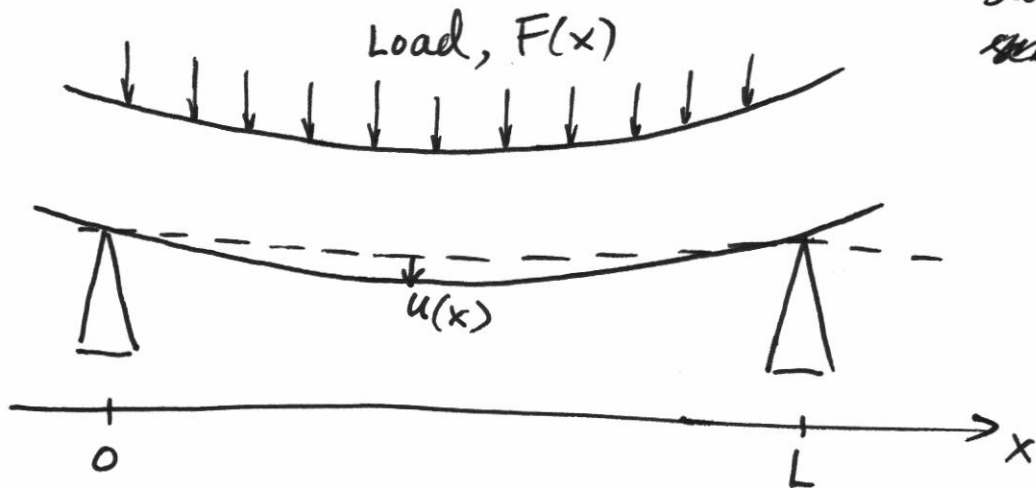


# FOURIER SERIES: The rigid beam problem

Butkov §4.8  
~~see~~ Example 2.



$$\frac{d^4 u}{dx^4} = \frac{1}{EI} F(x) \quad \text{for } 0 \leq x \leq L$$

Subtleties of the problem:

- Only interested in  $0 \leq x \leq L$ .
- Supports at  $x=0$  &  $x=L$  mean the ~~net load~~ <sup>beam doesn't bend</sup> ~~goes to zero~~ at these points.

The Fourier series for  $F(x)$  ~~should reflect this~~ <sup>can go to zero at  $x=0$</sup>   $x=L$ .

- A 4<sup>th</sup> Order ODE needs four constraints.

$$\text{In this case } u(0) = 0, \quad u(L) = 0$$

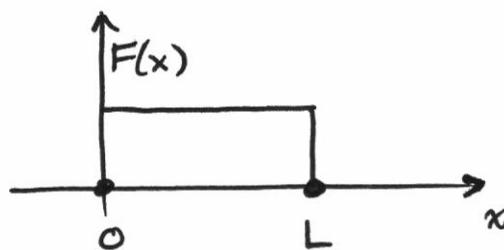
$$u''(0) = 0, \quad u''(L) = 0$$

No curvature at the points of support.

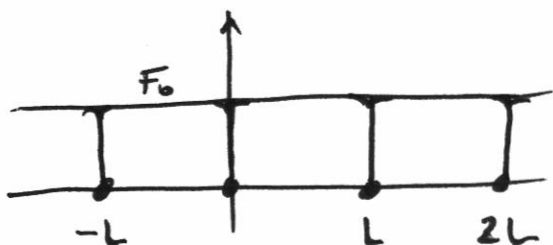
The Fourier series for  $u(x)$  must respect the 4 constraints.

(2)

What choices do we have for the Fourier Series?



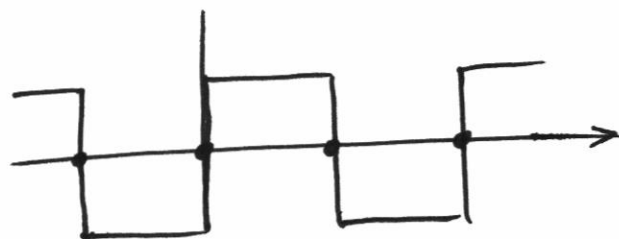
TURN INTO A PERIODIC FUNCTION



This choice is not writable as a Fourier series.

See §4.6 Butkov

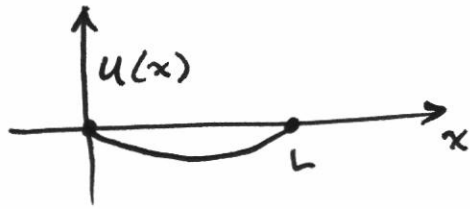
"Convergence of Fourier Series"



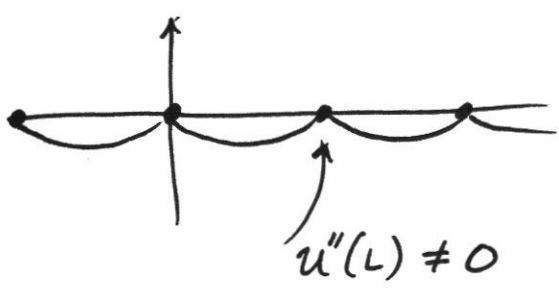
This choice can be written as a Fourier series.

$$F(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4F_0}{n\pi} \sin \frac{n\pi x}{L}$$

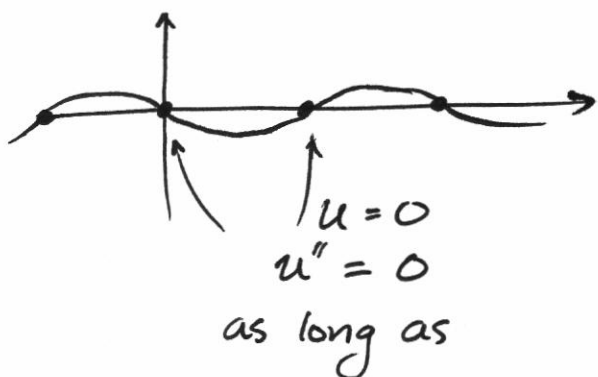
3



TURN INTO A PERIODIC FUNCTION



Does not respect constraint on  $u(x)$ .



$$u(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

This form will describe any continuous  $f_n$  of  $x$  between  $0 \leftrightarrow L$  that satisfies the 4 boundary condition constraints.

(4)

The D.E. can now be written as

$$u''''(x) = \frac{1}{EI} F(x)$$

$$\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^4 b_n \sin \frac{n\pi x}{L} = \frac{1}{EI} \sum_{n=1,3,5,\dots} \frac{4F_0}{n\pi} \sin \frac{n\pi x}{L}$$

where the  $b_n$  have not been determined yet.

By orthogonality of sin functions,

~~By~~

$$\left(\frac{n\pi}{L}\right)^4 b_n = \frac{4F_0}{\pi EI} \Rightarrow b_n = \left(\frac{L}{\pi}\right)^4 \frac{4F_0}{\pi EI}$$

$$\left(\frac{3\pi}{L}\right)^4 b_3 = \frac{4F_0}{3\pi EI} \Rightarrow b_3 = \left(\frac{L}{3\pi}\right)^4 \frac{4F_0}{3\pi EI}$$

$$\frac{b_3}{b_1} = \frac{1}{3^5} = \frac{1}{243}$$

The DE is essentially solved by the first term in the Fourier series. Corrections are of order 0.4%.

Fourier series works well here because the answer converges quickly to ~~an~~ get high accuracy.

(5)

# THE FOURIER COSINE TRANSFORM

Adapted from Butkov §7.1.

Start from the Fourier cosine series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad [\text{series}]$$

where  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

even fn over interval  $-L \leftrightarrow L$

Notice that  $\frac{n\pi}{L}$  keeps showing up.

Give this quantity a name.

$$\frac{n\pi}{L} = k \quad \text{"wavenumber"}$$

Wave number tells us how quickly a wave oscillates thru space.

As  $L \rightarrow \infty$ , the wave numbers become closely spaced  $k = 0, \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \dots$

$$\sum_{n=0}^{\infty} \rightarrow \int_{k=0}^{\infty} \frac{dk}{\frac{\pi}{L}}$$

$\frac{\pi}{L}$  is the spacing between  $k$ .

(6)

Changing the series into an integral yields

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_{n=\frac{k}{\pi/L}} \cos(kx) \frac{dk}{\pi/L} \\ &= \frac{1}{\pi} \int_0^{\infty} L a_{\frac{kL}{\pi}} \cos(kx) dk \end{aligned}$$

where

$$L a_{\frac{kL}{\pi}} = \lim_{L \rightarrow \infty} \int_{-L}^L f(x) \cos(kx) dx$$

Note that the quantity  $L a_{\frac{kL}{\pi}}$  is well behaved as  $L \rightarrow \infty$ , unlike  $a_n$  which goes to zero as  $L \rightarrow \infty$ .

Butkov defines the transform (§7.6) as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(k) \cos kx dx$$

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos kx dx$$

(7)  
The transform gives us a second way  
of representing a function



Both representations contain  
all the information about the function.  
 $f(x)$  can be constructed from  $F_c(k)$   
and vice versa.

Exercise: Calculate  $F_c(k)$  for the top hat  
function shown above.