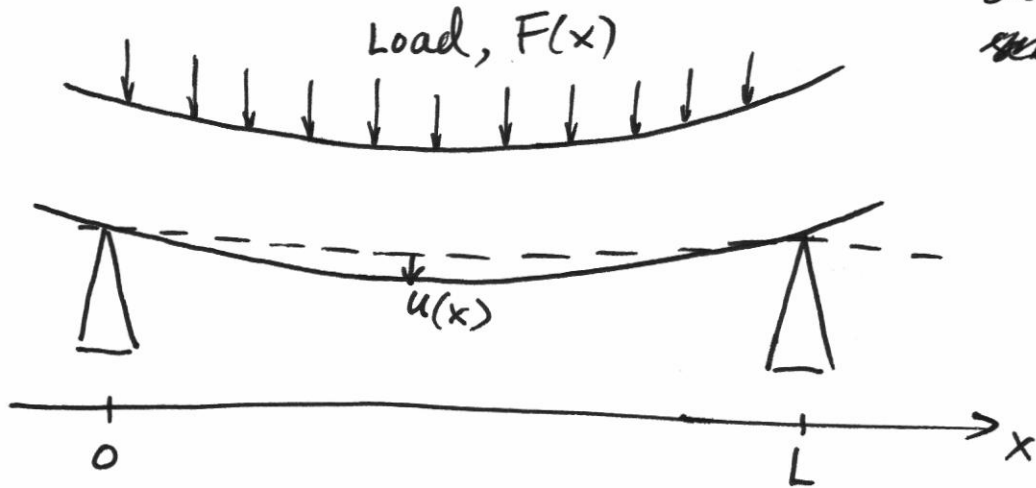


FOURIER SERIES: The rigid beam problem

Butkov §4.8
~~see~~ Example 2.



$$\frac{d^4 u}{dx^4} = \frac{1}{EI} F(x) \quad \text{for } 0 \leq x \leq L$$

Subtleties of the problem:

- Only interested in $0 \leq x \leq L$.
- Supports at $x=0$ & $x=L$ mean the ~~net load~~ ^{beam doesn't bend} ~~goes to zero~~ at these points.

The Fourier series for $F(x)$ ~~should reflect this~~ ^{can go to zero at $x=0$} $x=L$.

- A 4th Order ODE needs four constraints.

$$\text{In this case } u(0) = 0, \quad u(L) = 0$$

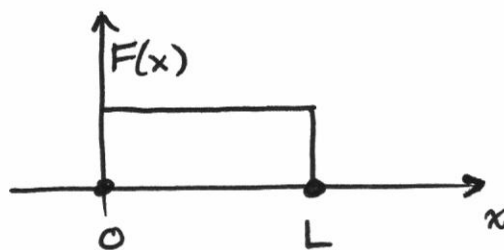
$$u''(0) = 0, \quad u''(L) = 0$$

No curvature at the points of support.

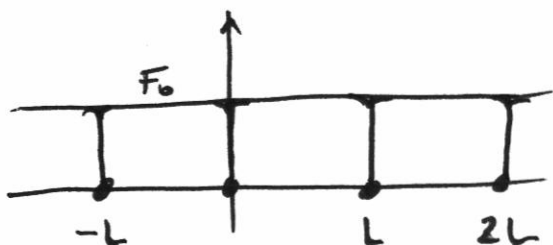
The Fourier series for $u(x)$ must respect the 4 constraints.

(2)

What choices do we have for the Fourier Series?



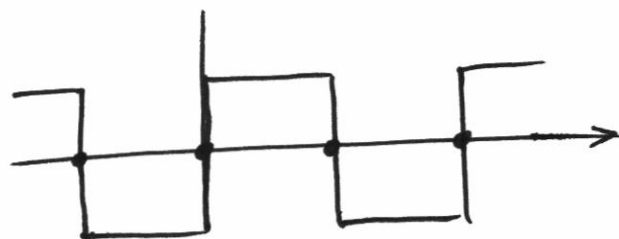
TURN INTO A PERIODIC FUNCTION



This choice is not writable as a Fourier series.

See §4.6 Butkov

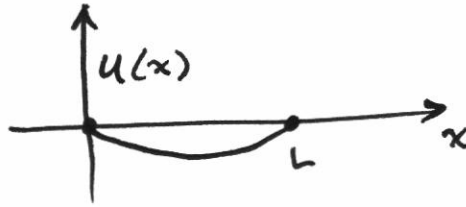
"Convergence of Fourier Series"



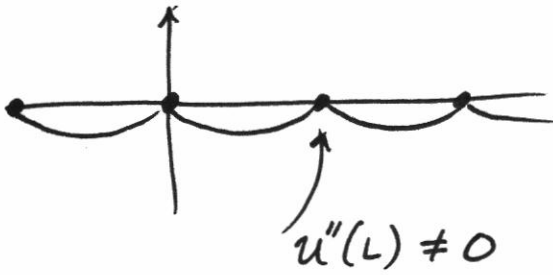
This choice can be written as a Fourier series.

$$F(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4F_0}{n\pi} \sin \frac{n\pi x}{L}$$

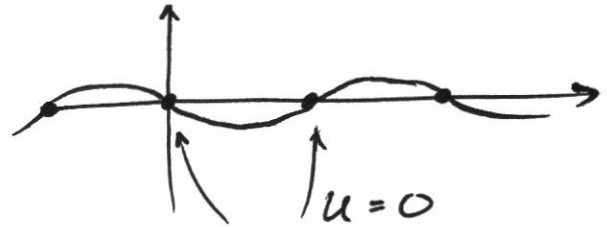
(3)



TURN INTO A PERIODIC FUNCTION



Does not respect constraint on $u(x)$.



as long as

$$u(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

This form will describe any continuous fn of x between $0 \leftrightarrow L$ that satisfies the 4 boundary condition constraints.

(4)

The D.E. can now be written as

$$u''''(x) = \frac{1}{EI} F(x)$$

$$\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^4 b_n \sin \frac{n\pi x}{L} = \frac{1}{EI} \sum_{n=1,3,5,\dots} \frac{4F_0}{n\pi} \sin \frac{n\pi x}{L}$$

where the b_n have not been determined yet.

By orthogonality of sin functions,

~~By~~

$$\left(\frac{n\pi}{L}\right)^4 b_n = \frac{4F_0}{\pi EI} \Rightarrow b_n = \left(\frac{L}{\pi}\right)^4 \frac{4F_0}{\pi EI}$$

$$\left(\frac{3\pi}{L}\right)^4 b_3 = \frac{4F_0}{3\pi EI} \Rightarrow b_3 = \left(\frac{L}{3\pi}\right)^4 \frac{4F_0}{3\pi EI}$$

$$\frac{b_3}{b_1} = \frac{1}{3^5} = \frac{1}{243}$$

The DE is essentially solved by the first term in the Fourier series. Corrections are of order 0.4%.

Fourier series works well here because the answer converges quickly to ~~an~~ get high accuracy.

(5)

THE FOURIER COSINE TRANSFORM

Adapted from Butkov §7.1.

Start from the Fourier cosine series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad [\text{series}]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

even f_n over interval $-L \leftrightarrow L$

Notice that $\frac{n\pi}{L}$ keeps showing up.

Give this quantity a name.

$$\frac{n\pi}{L} = k \quad \text{"wavenumber"}$$

Wave number tells us how quickly a wave oscillates thru space.

As $L \rightarrow \infty$, the wave numbers become closely spaced $k = 0, \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \dots$

$$\sum_{n=0}^{\infty} \rightarrow \int_{k=0}^{\infty} \frac{dk}{\frac{\pi}{L}}$$

$\frac{\pi}{L}$ is the spacing between k .

(6)

Changing the series into an integral yields

$$f(x) = \int_{k=0}^{\infty} a_{n=\frac{k}{\pi/L}} \cos(kx) \frac{dk}{\pi/L}$$

$$= \frac{1}{\pi} \int_0^{\infty} L a_{\frac{kL}{\pi}} \cos(kx) dk$$

where

$$L a_{\frac{kL}{\pi}} = \lim_{L \rightarrow \infty} \int_{-L}^L f(x) \cos(kx) dx$$

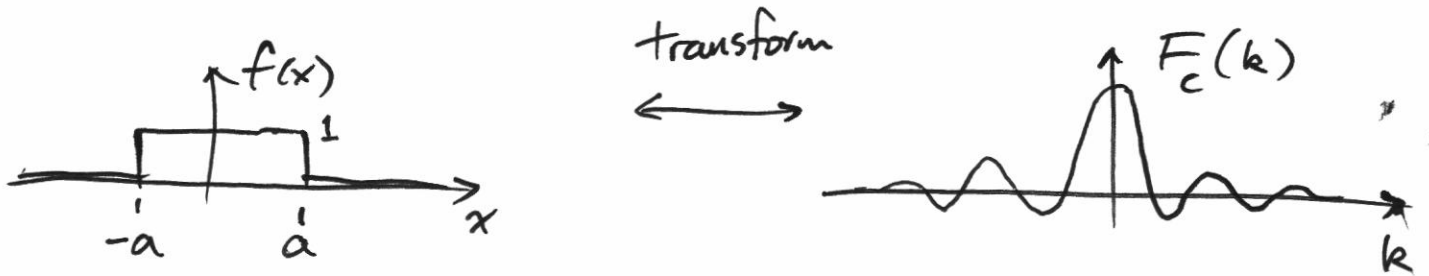
Note that the quantity $L a_{\frac{kL}{\pi}}$ is well behaved as $L \rightarrow \infty$, unlike a_n which goes to zero as $L \rightarrow \infty$.

Butkov defines the transform (§7.6) as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(k) \cos kx dx$$

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos kx dx$$

(7)
The transform gives us a second way
of representing a function



Both representations contain
all the information about the function.
 $f(x)$ can be constructed from $F_c(k)$
and vice versa.

Exercise: Calculate $F_c(k)$ for the top hat
function shown above.