

FOURIER SERIES

A tool that should be (or become) second nature to you.

§4.1-4.4, 4.8 Butkov

Typically used for periodic functions of time
or periodic functions of space.

(Notation slightly different for $f(t)$ vs. $f(x)$.)

DEFINITION:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right)$$

OR

[Period T]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

[Period $2L$]

where

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad n > 0$$

The formulas for a_n & b_n are based on the orthogonality of trig fns.

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For example, I want to find the coefficient for $\cos \frac{4\pi x}{L}$ in my periodic function $f(x)$.

Formula says:

$$a_4 = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{4\pi x}{L} dx$$

show that RHS = a_4

$$\text{RHS} = \frac{1}{L} \int_{-L}^L \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \cos \frac{4\pi x}{L} dx$$

First term $\int_{-L}^L \frac{a_0}{2} \cos \frac{4\pi x}{L} dx =$



Next terms $\int_{-L}^L a_n \cos \frac{\pi x}{L} \cos \frac{4\pi x}{L} dx =$

$$\int_{-L}^L b_n \sin \frac{\pi x}{L} \cos \frac{4\pi x}{L} dx =$$

To evaluate these ^{next} terms, some trig identities are useful.

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$$\cos nx \cos mx$$

$$= \frac{1}{2}(e^{inx} + e^{-inx}) \frac{1}{2}(e^{imx} + e^{-imx})$$

$$= \frac{1}{4} \left(e^{i(n+m)x} + e^{-i(n+m)x} \right) \left(e^{i(n-m)x} + e^{-i(n-m)x} \right)$$

$$= \frac{1}{2} \cos(n+m)x + \frac{1}{2} \cos(n-m)x$$

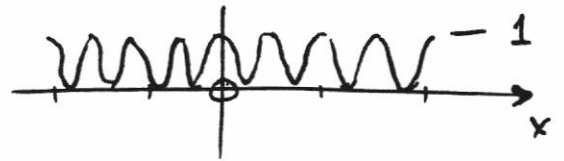
[Frequency mixing leads to sum & difference freqs]

$$\rightarrow \int_{-L}^L a_1 \cos \frac{\pi x}{L} \cos \frac{4\pi x}{L} dx$$

$$= \int_{-L}^L \frac{1}{2} \cos 5x dx + \int_{-L}^L \frac{1}{2} \cos 3x dx$$

$$= 0$$

$$\text{RHS} = \frac{1}{L} \int_{-L}^L a_4 \cos^2 \frac{4\pi x}{L} dx$$



$$= \frac{1}{L} 2L \frac{1}{2} a_4$$

$$= a_4$$

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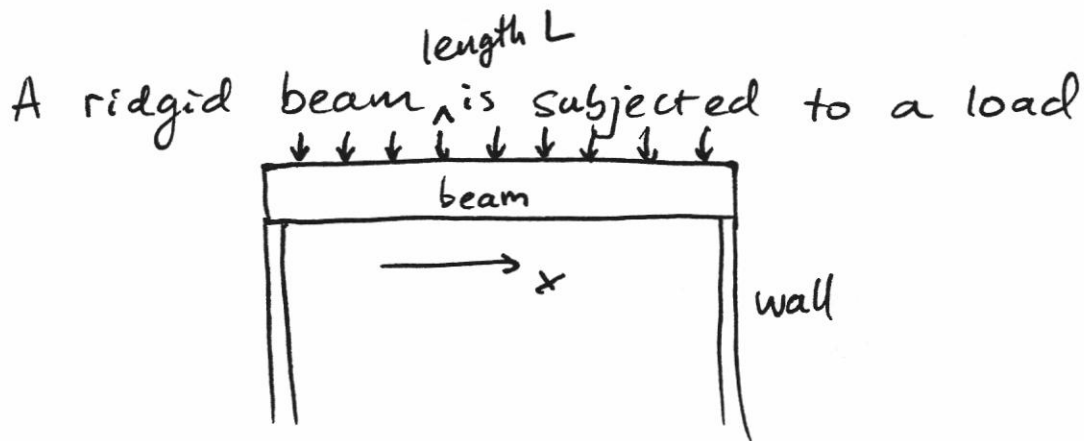
Exercise: Find Fourier series for

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ +1 & 0 < x < \pi \end{cases}$$

$$f(x) = \sum_{n=1,3,5,\dots} \frac{4}{n\pi} \sin n x$$

One application of Fourier series:

(there are hundreds)



Vertical deflection of beam is $u(x)$.

$$\frac{d^4 u}{dx^4} = \frac{1}{EI} F(x)$$

load per unit length.

We did a similar problem for a stretched string.
(What technique?)

Choose an easy case: Uniform load $F(x) = F_0$

Both the LHS & RHS are amenable to a Fourier series expansion.

$$u(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$F(x) = \sum_{n=1,3,5} \frac{4F_0}{n\pi} \sin \frac{n\pi x}{L}$$

$$\frac{d^4 u}{dx^4} = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^4 b_n \sin \frac{n\pi x}{L}$$

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The D.E. can now be written as

$$\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^4 b_n \sin \frac{n\pi x}{L} = \frac{1}{EI} \sum_{n=1,3,5,\dots} \frac{4F_0}{n\pi} \sin \frac{n\pi x}{L}$$

where the b_n
have not been
determined yet.

By orthogonality of sin functions,

~~By~~

$$\left(\frac{n\pi}{L}\right)^4 b_n = \frac{4F_0}{\pi EI} \Rightarrow b_n = \left(\frac{L}{\pi}\right)^4 \frac{4F_0}{\pi EI}$$

$$\left(\frac{3\pi}{L}\right)^4 b_3 = \frac{4F_0}{3\pi EI} \Rightarrow b_3 = \left(\frac{L}{3\pi}\right)^4 \frac{4F_0}{3\pi EI}$$

$$\frac{b_3}{b_1} = \frac{1}{3^5} = \frac{1}{243}$$

The DE is essentially solved by the first term in the Fourier series. Corrections are of order 0.4%.

Fourier series works well here because the answer converges quickly to ~~an~~ get high accuracy.