FOURIER SERIES

A tool that should be (or become) second nature to you.

§4.1-4.4, 4.8 Butkov

Typically used for periodic functions of time or periodic functions of space.

(Notation slightly different for $f(t)$ vs. $f(x)$)

DEFINITION:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi}{T} nt + b_n \sin \frac{2\pi}{T} nt \right)$$

[Period T]

OR

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

[Period 2L]

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \qquad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \qquad n > 0$$

The formulas for a_n & b_n are based on the orthogonality of trig fns.

For example, I want to find the coefficient for $\cos \frac{4\pi x}{L}$ in my periodic function $f(x)$.

Formula says:

$$a_4 = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{4\pi x}{L}\, dx$$

show that RHS $= a_4$

$$\text{RHS} = \frac{1}{L}\int_{-L}^{L} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \cos \frac{4\pi x}{L}\, dx$$

First term $\displaystyle\int_{-L}^{L} \frac{a_0}{2} \cos \frac{4\pi x}{L}\, dx =$

Next terms $\displaystyle\int_{-L}^{L} a_1 \cos \frac{\pi x}{L} \cos \frac{4\pi x}{L}\, dx =$

$$\int_{-L}^{L} b_1 \sin \frac{\pi x}{L} \cos \frac{4\pi x}{L}\, dx =$$

To evaluate these next terms, some trig identities are useful.

$$\cos nx \; \cos mx$$

$$= \frac{1}{2}\left(e^{inx} + e^{-i\cancel{m}nx}\right)\frac{1}{2}\left(e^{imx} + e^{-imx}\right)$$

$$= \frac{1}{4}\left(e^{i(n+m)x} + e^{-i(n+m)x}\right)\left(e^{i(n-m)x} + e^{-i(n-m)x}\right)$$

$$= \frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x$$

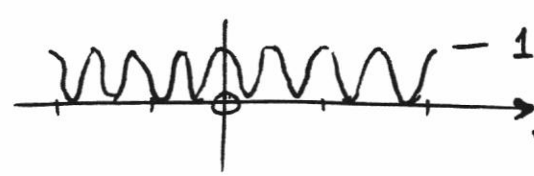
[Frequency mixing leads to sum & difference freqs]

$$\longrightarrow \quad \int_{-L}^{L} a_1 \cos\frac{\pi x}{L}\cos\frac{4\pi x}{L}\,dx$$

$$= \int_{-L}^{L} \frac{1}{2}\cos 5x\,dx + \int_{-L}^{L}\frac{1}{2}\cos 3x\,dx$$

$$= 0$$

$$\text{RHS} = \frac{1}{L}\int_{-L}^{L} a_4 \cos^2\frac{4\pi x}{L}\,dx$$



$$= \frac{1}{L}\,2L\,\frac{1}{2}\,a_4$$

$$= a_4$$

Exercise: Find Fourier series for

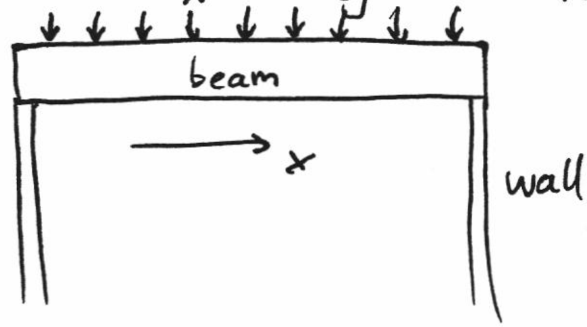
$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ +1 & 0 < x < \pi \end{cases}$$

$$f(x) = \sum_{n=1,3,5\ldots} \frac{4}{n\pi} \sin n x$$

One application of Fourier series:

(there are hundreds)

A ridgid beam length L is subjected to a load



Vertical deflection of beam is $u(x)$.

$$\frac{d^4u}{dx^4} = \frac{1}{EI}F(x)$$

load per unit length.

We did a similar problem for a stretched string.
(What technique?)

Choose an easy case: Uniform load $F(x) = F_0$

Both the LHS & RHS are amenable to a Fourier series expansion.

$$u(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad F(x) = \sum_{n=1,3,5} \frac{4F_0}{n\pi}\sin\frac{n\pi x}{L}$$

$$\frac{d^4u}{dx^4} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^4 b_n \sin\frac{n\pi x}{L}$$

The D.E. can now be written as

$$\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^4 b_n \sin\frac{n\pi x}{L} = \frac{1}{EI} \sum_{n=1,3,5\ldots} \frac{4F_0}{n\pi} \sin\frac{n\pi x}{L}$$

where the b_n have not been determined yet.

By orthogonality of sin functions,

$$\left(\frac{n\pi}{L}\right)^4 b_1 = \frac{4F_0}{\pi EI} \quad \Rightarrow \quad b_1 = \left(\frac{L}{\pi}\right)^4 \frac{4F_0}{\pi EI}$$

$$\left(\frac{3\pi}{L}\right)^4 b_3 = \frac{4F_0}{3\pi EI} \quad \Rightarrow \quad b_3 = \left(\frac{L}{3\pi}\right)^4 \frac{4F_0}{3\pi EI}$$

$$\frac{b_3}{b_1} = \frac{1}{3^5} = \frac{1}{243}$$

The DE is essentially solved by the first term in the Fourier series. Corrections are of order 0.4%.

Fourier series works well here because the answer converges quickly to ~~an~~ get high accuracy.