Last time: Residue Thm:

$$\oint_C f(z)\,dz = 2\pi i \sum_j \text{Res}\, f(z_j)$$

Today: Practice applying residue thm to calculate definite integrals.

Buthov has several nice examples in §2.12.

I'll use examples ~~[illegible]~~ from 24.13.1 & 24.13.2 Riley.

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2+b^2-2ab\cos\theta}\,d\theta \quad , \quad b>a>0$$

Note: This is a common expression coming from $|\vec{a}-\vec{b}|^2$

Step 1: Recast the integrand in terms of a complex number $z = e^{i\theta}$.

$$\cos\theta = \tfrac{1}{2}(z+z^{-1})$$

$$\cos 2\theta = \tfrac{1}{2}(z^2+z^{-2})$$

$$\text{Integrand} = \frac{\frac{1}{2}\left(z^2 + z^{-2}\right)}{a^2+b^2 - ab\left(z + z^{-1}\right)}$$

Step 2: Indentify poles ~~in the integrand~~.
This is best done by factorizing the numerator and the denominator

$$\text{Integrand} = \frac{\frac{1}{2z^2}\left(z^4 + 1\right)}{\frac{ab}{z}\left(\frac{a}{b}z + \frac{b}{a}z - z^2 - 1\right)}$$

$$= -\frac{z}{ab}\frac{1}{2z^2}\frac{\left(z^4+1\right)}{\left(z - \frac{a}{b}\right)\left(z - \frac{b}{a}\right)}$$

$$= \frac{-1}{2ab}\frac{\left(z^4+1\right)}{z\left(z - \frac{a}{b}\right)\left(z - b/a\right)}$$

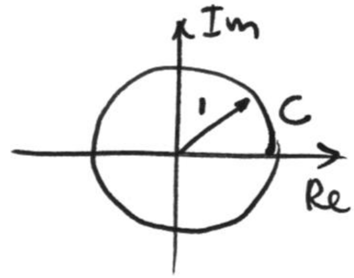
The full integral is then

$$I = \oint_C \frac{-1}{2ab}\frac{z^4+1}{z\left(z-\frac{a}{b}\right)\left(z-\frac{b}{a}\right)}\frac{dz}{iz}$$

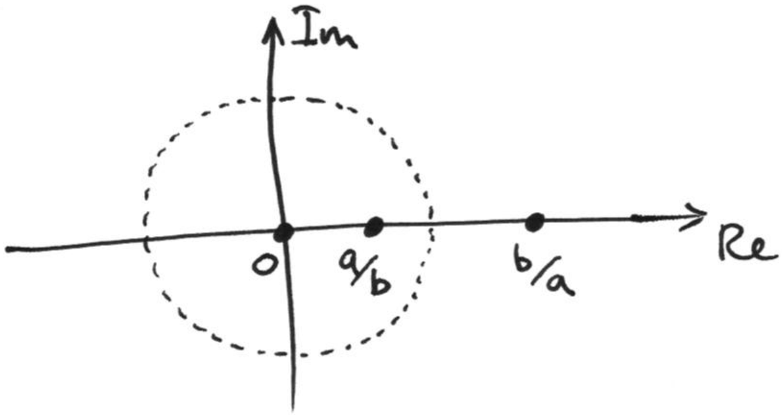
where I used $dz = ie^{i\theta}d\theta$
$= iz\,d\theta$

(see notes from Day 10)

$$I = \frac{i}{2ab} \oint_C \frac{z^4 + 1}{z^2 \left(z - \frac{a}{b}\right)\left(z - \frac{b}{a}\right)} \, dz$$



There are two poles inside C (recall $b > a > 0$)



$z = 0$ is a pole of order 2

$\lim_{z \to 0} z f(z)$ does not exist ($m \neq 1$)

$\lim_{z \to 0} z^2 f(z)$ exists, $m = 2$ ✓

$z = \frac{a}{b}$ is a pole of order 1

$\lim_{z \to \frac{a}{b}} \left(z - \frac{a}{b}\right) f(z)$ exists, $m = 1$ ✓

$\boxed{\text{FINAL STEP}}$ Find $\operatorname{Res} f(0)$ & $\operatorname{Res} f\left(\frac{a}{b}\right)$

$$\operatorname{Res} f(0) = \frac{1}{(m-1)!} \lim_{z\to 0} \left\{ \frac{d^{m-1}}{dz^{m-1}} z^m f(z) \right\} \quad \text{where } m=2$$

$$= \lim_{z\to 0} \left\{ \frac{d}{dz} z^2 f(z) \right\}$$

$$= \lim_{z\to 0} \left\{ \frac{d}{dz} \frac{z^4+1}{(z-a/b)(z-b/a)} \right\}$$

$$= \lim_{z\to 0} \left\{ \frac{4z^3}{(z-\frac{a}{b})(z-b/a)} - \frac{z^4+1}{(z-\frac{a}{b})^2(z-\frac{b}{a})} - \frac{z^4+1}{(z-\frac{a}{b})(z-\frac{b}{a})^2} \right\}$$

$$= \frac{-1}{-\frac{a^2 b}{b^2 a}} - \frac{1}{\frac{-ab^2}{ba^2}} = \frac{b}{a} + \frac{a}{b}$$

$$\operatorname{Res} f\left(\frac{a}{b}\right) = \lim_{z\to \frac{a}{b}} \left(z-\frac{a}{b}\right) \frac{(z^4+1)}{z^2(z-\frac{a}{b})(z-\frac{b}{a})}$$

$$= \frac{\frac{a^4}{b^4}+1}{\frac{a^2}{b^2}\left(\frac{a}{b}-\frac{b}{a}\right)} = \frac{a^4+b^4}{a^2b^2\left(\frac{a}{b}-\frac{b}{a}\right)}$$

$$= \frac{a^4+b^4}{ab(a^2-b^2)}$$

Applying residue theorem

$$I = 2\pi i \left(\operatorname{Res} f(0) + \operatorname{Res} f\left(\frac{a}{b}\right) \right)$$

$$= 2\pi i \, \frac{i}{2ab} \left(\frac{b}{a} + \frac{a}{b} + \frac{a^4 + b^4}{ab(a^2 - b^2)} \right)$$

$$= \quad \Big\downarrow \text{ skip 4 lines of algebra}$$

$$= \frac{2\pi a^2}{b^2(b^2 - a^2)}$$

Your turn to try an example:

$$I = \int_0^\infty \frac{dx}{(x^2 + a^2)^2} \,, \quad \text{evaluate } I.$$