DAY 9

PH 424

WHITEBOARD: Write down an eigenvalue eqn.

Summary from yesterdays worksheet

	$\psi(x) = Ae^{ikx}$	$= Ae^{-ikx}$	$= A \sin kx$
$\hat{p} \longrightarrow i\hbar \frac{\partial}{\partial x}$	Eigenvalue of $\hat{p}$ is k	$-k$	not an eigenfn.
$\frac{\hat{p}^2}{2m} \rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$	Eigenvalue. $\frac{\hbar^2 k^2}{2m}$	$\frac{\hbar^2 k^2}{2m}$	$\frac{\hbar^2 k^2}{2m}$

There are many eigenvalue eqns in physics, but in this course we will refer to one eigenvalue eqn as "The Eigenvalue Eqn".

$$\text{S. Eq}^n \xrightarrow[\text{of variables}]{\text{separation}} \text{The Eigenvalue Eq}^n.$$

$$\hat{H}\,\Psi(x,t) = i\hbar \frac{\partial}{\partial t}\Psi(x,t)$$

let $\Psi(x,t) = \phi(x)T(t)$

then $$\hat{H}\,\phi(x)T(t) = i\hbar \frac{\partial}{\partial t}\phi(x)T(t)$$

$$T(t)\,\hat{H}\,\phi(x) = i\hbar\,\phi(x)\frac{dT(t)}{dt}$$

$$\frac{1}{\phi(x)}\hat{H}\,\phi(x) = i\hbar\frac{1}{T(t)}\frac{dT(t)}{dt}$$

$$LHS = RHS = \text{const} = E$$

The separation const.

$$\frac{1}{\phi(x)}\hat{H}\,\phi(x) = E$$

This will be our famous eigenvalue eqn

$$\boxed{\hat{H}\,\phi_E(x) = E\,\phi_E(x)}$$

The subscript E reminds us that a different eigenfn exists for each eigenvalue.

The most general solution to the S. Eqn will be

$$\Psi(x,t) = \sum_i c_i\,\phi_{E_i}(x)T_{E_i}(t)$$

Finding the set of $\phi_E(x)$ functions is a central theme of quantum mechanics.

To find $\phi_E(x)$:

$\boxed{\text{STEP 1}}$ Write $\hat{H}$ in differential form.

Typically K.E. and P.E. are the main terms in the Hamiltonian

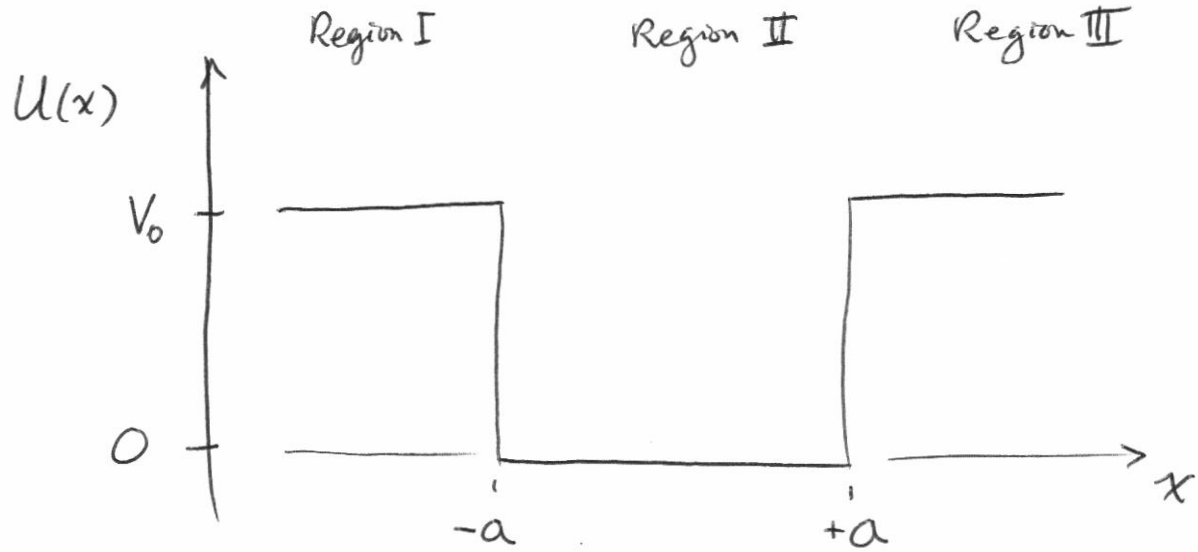
i.e. $$\hat{H} = \frac{\hat{p}^2}{2m} + U(x)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)$$

This potential energy changes from problem to problem.

$\boxed{\text{STEP 2}}$ Wrestle with the eigenvalue eqn written as a differential eqn.

Small Group Activity



Solution:

The potential $U(x)$ is a piecewise fn, so we will solve the eigenvalue eqn in a piecewise manner.

Region I

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right)\phi_E(x) = E\phi_E(x)$$

Region III

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right)\phi_E(x) = E\phi_E(x)$$

Region II

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi_E(x) = E\phi_E(x)$$

Region I & III

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_E}{dx^2} = (E - V_0)\,\phi_E(x)$$

$$\frac{d^2\phi_E}{dx^2} = \underbrace{\frac{2m}{\hbar^2}(V_0 - E)}_{\text{positive const.}}\,\phi_E \qquad \Rightarrow \quad \phi_E(x) = A e^{k_1 x} + B e^{-k_1 x}$$

(two linearly indep ~~sols~~ terms means this is a general solution)

Region II

$$\cancel{-\frac{\hbar^2}{2m}}\;\frac{d^2\phi_E}{dx^2} = \underbrace{-\frac{2m}{\hbar^2}(V_0 - E)}_{\text{Negative const}}\,\phi_E \qquad \Rightarrow \quad \phi_E(x) = C e^{ik_2 x} + D e^{-ik_2 x}$$

Now we must find specific solutions (i.e. specific values of A, B, C, D etc.) that ~~satisfy~~ give a physically reasonable $\phi_E(x)$.

- $\phi_E(x)$ must be continuous across the regions.
- $\dfrac{d\phi_E(x)}{dx}$ must be continuous across the regions.
- If $E < V_0$, ϕ_E must go to zero when $|x| \gg a$.