

DAY 9

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PH 424

WHITEBOARD: write down an eigenvalue eqⁿ.

Summary from yesterday's worksheet

	$\psi(x) = Ae^{ikx}$	$= Ae^{-ikx}$	$= A \sin kx$
$\hat{p} \rightarrow i\hbar \frac{\partial}{\partial x}$	Eigenvalue of \hat{p} is k	$-k$	not an eigenfn.
$\frac{\hat{p}^2}{2m} \rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$	Eigenvalue. $\frac{\hbar^2 k^2}{2m}$	$\frac{\hbar^2 k^2}{2m}$	$\frac{\hbar^2 k^2}{2m}$

There are many eigenvalue eqns in physics, but in this course we will refer to one eigenvalue eqn as "The Eigenvalue Eqn".

S. Eqn

separation of variables

The Eigenvalue Eqn.

(2)

$$\hat{H} \Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t)$$

let $\Psi(x, t) = \phi(x)T(t)$

then $\hat{H} \phi(x)T(t) = i\hbar \frac{\partial}{\partial t} \phi(x)T(t)$

$$T(t) \hat{H} \phi(x) = i\hbar \phi(x) \frac{dT(t)}{dt}$$

$$\frac{1}{\phi(x)} \hat{H} \phi(x) = i\hbar \frac{1}{T(t)} \frac{dT(t)}{dt}$$

$$\text{LHS} = \text{RHS} = \text{const} = E$$

↑
The separation const.

$$\frac{1}{\phi(x)} \hat{H} \phi(x) = E$$

This will be our famous
eigenvalue eqⁿ

$$\hat{H} \phi_E(x) = E \phi_E(x)$$

The subscript E
reminds us that
a different eigenfn
exists for each eigenvalue.

The most general solution to the S. Eqⁿ will be

$$\Psi(x, t) = \sum_i c_i \phi_{E_i}(x) T_{E_i}(t)$$

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Finding the set of $\phi_E(x)$ functions is a central theme of quantum mechanics.

To find $\phi_E(x)$:

STEP 1 Write \hat{H} in differential form.

Typically K.E. and P.E. are the main terms in the Hamiltonian

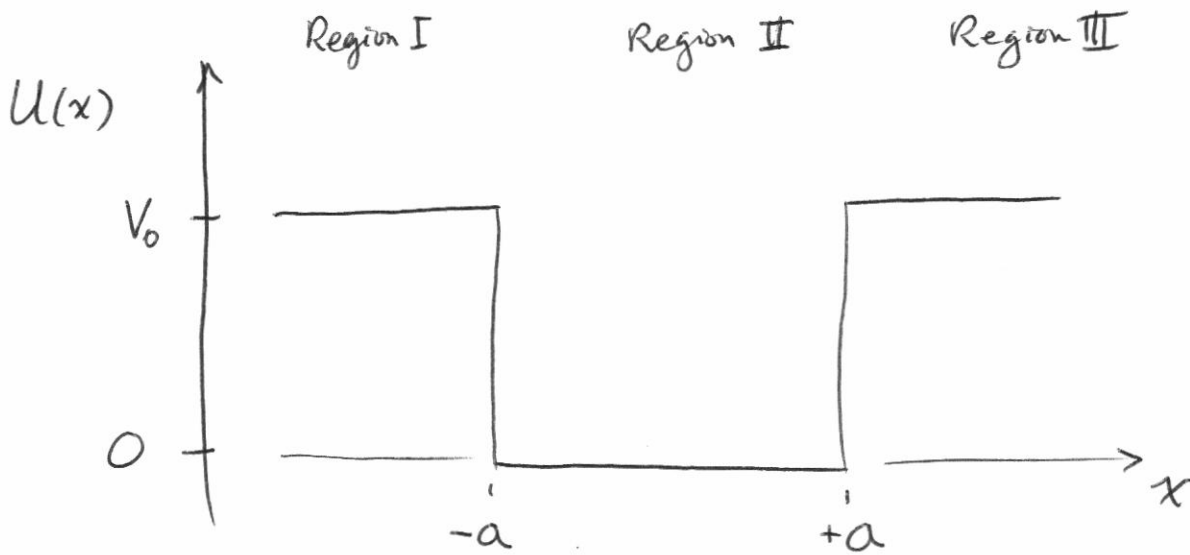
$$\begin{aligned} \text{i.e. } \hat{H} &= \frac{\hat{p}^2}{2m} + U(x) \\ &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) \end{aligned}$$

↖ This potential energy changes from problem to problem.

STEP 2 Wrestle with the eigenvalue eqn written as a differential eqn.

(4)

Small Group Activity



Solution:

The potential $U(x)$ is a piecewise fn, so we will solve the eigenvalue eqn in a piecewise manner.

Region I

$$\left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V_0 \right) \phi_E(x) = E \phi_E(x)$$

Region III

$$\left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V_0 \right) \phi_E(x) = E \phi_E(x)$$

Region II

$$\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \phi_E(x) = E \phi_E(x)$$

(5)

Region I & III

$$-\frac{\hbar^2}{2m} \frac{d^2 \phi_E}{dx^2} = (E - V_0) \phi_E(x)$$

$$\frac{d^2 \phi_E}{dx^2} = \underbrace{\frac{2m(V_0 - E)}{\hbar^2}}_{\text{positive const.}} \phi_E$$

$$\Rightarrow \phi_E(x) = A e^{k_1 x} + B e^{-k_1 x}$$

(two linearly indep ~~set~~ terms means this is a general solution)

Region II

$$\frac{d^2 \phi_E}{dx^2} = \underbrace{-\frac{2m(V_0 - E)}{\hbar^2}}_{\text{Negative const}} \phi_E$$

$$\Rightarrow \phi_E(x) = C e^{i k_2 x} + D e^{-i k_2 x}$$

Now we must find specific solutions (i.e. specific values of A, B, C, D etc.) that ~~satisfy~~ give a physically reasonable $\phi_E(x)$.

- $\phi_E(x)$ must be continuous across the regions.
- $\frac{d\phi_E(x)}{dx}$ must be continuous across the regions.
- If $E < V_0$, ϕ_E must go to zero when $|x| \gg a$.