

Consider the following wave state which describes an electron in an infinite potential well.

$$|\Phi\rangle = \sqrt{\frac{1}{5}} |\phi_1\rangle + \sqrt{\frac{3}{5}} |\phi_2\rangle + \sqrt{\frac{1}{5}} |\phi_3\rangle$$

which can be written in wave fn notation ~~as~~ as

$$\Phi(x) @t=0 = \begin{cases} 0 & \text{I} \\ \sqrt{\frac{1}{5}} \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} + \sqrt{\frac{3}{5}} \sqrt{\frac{2}{L}} \sin \frac{2\pi x}{L} + \sqrt{\frac{1}{5}} \sqrt{\frac{2}{L}} \sin \frac{3\pi x}{L} & \text{II} \\ 0 & \text{III} \end{cases}$$

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Is this an energy eigenstate?

**Answer** No, it does not satisfy the eigenvalue eqn

$$H|\Phi\rangle \neq \text{const} |\Phi\rangle$$

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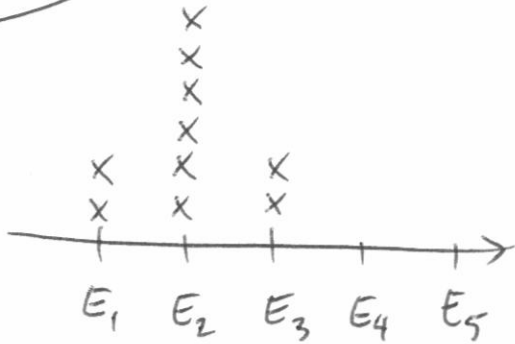
What are the possible energies I could measure?

**Answer**  $E_1$ ,  $E_2$  &  $E_3$

②

If I make lots of measurements<sub>n</sub> of this state  
(always preparing the state exactly the same way)  
what will be the distribution of results?

Answer



$$P_{E_1} = 1/5$$

$$P_{E_2} = 3/5$$

$$P_{E_3} = 1/5$$

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What is  $\langle E \rangle$ ? i.e. calculate  $\langle E \rangle$ .

Answer

$$\langle \Phi | H | \Phi \rangle =$$

$$= \langle \Phi | H \left( \sqrt{\frac{1}{5}} |\phi_1\rangle + \sqrt{\frac{3}{5}} |\phi_2\rangle + \sqrt{\frac{1}{5}} |\phi_3\rangle \right)$$

$$= \langle \Phi | \left( \sqrt{\frac{1}{5}} E_1 |\phi_1\rangle + \sqrt{\frac{3}{5}} E_2 |\phi_2\rangle + \sqrt{\frac{1}{5}} E_3 |\phi_3\rangle \right)$$

$$= \frac{1}{5} E_1 + \frac{3}{5} E_2 + \frac{1}{5} E_3$$

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Calculate  $\Delta E$  ③

Answer

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$$

$$\langle E^2 \rangle = \langle \Phi | \hat{H}^2 | \Phi \rangle$$

$$= \frac{1}{5} E_1^2 + \frac{3}{5} E_2^2 + \frac{1}{5} E_3^2$$

Then work thru the algebra to find  $\Delta E$ .

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Note: These calculation can also be done ~~in~~ as wave function integrals. However, ket notation is typically more convenient. There are some calculations where you have no choice but to use wavefn integrals.

Example: Find the expectation value of position

$$\langle \hat{x} \rangle = \langle \Phi | \hat{x} | \Phi \rangle$$

$$= \int_{-\infty}^{\infty} \Phi^*(x) x \Phi(x) dx$$

↖ The operator used to measure position

(4)

Example: Find the expectation value of momentum

$$\langle \hat{p} \rangle = \langle \Phi | -i\hbar \frac{d}{dx} | \Phi \rangle$$

$$= \int_{-\infty}^{\infty} \Phi^*(x) \left( -i\hbar \frac{d}{dx} \right) \Phi(x) dx$$

$$= \int_0^L \Phi^*(x) \left( -i\hbar \frac{d}{dx} \right) \left( \sqrt{\frac{1}{5}} \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} + \sqrt{\frac{3}{5}} \sqrt{\frac{2}{L}} \sin \frac{2\pi x}{L} + \sqrt{\frac{1}{5}} \sqrt{\frac{2}{L}} \sin \frac{3\pi x}{L} \right) dx$$

You will end up with 9 cross terms

$$\int_0^L \sin \frac{\pi x}{L} \cos \frac{\pi x}{L} dx$$

$$\int_0^L \sin \frac{\pi x}{L} \cos \frac{2\pi x}{L} dx \quad \text{etc.}$$

Everything is multiplied by  $i$

$\langle \hat{p} \rangle$  cannot be imaginary, so the integrals must add up to zero.

For a physical argument: We know the electron is ~~not~~ not escaping to the left or right.

$$\Rightarrow \langle \hat{p} \rangle = 0.$$

(5)

# NOW ADD TIME DEPENDENCE

$$\hat{H}|\Psi\rangle = i\hbar \frac{\partial}{\partial t} |\Psi\rangle$$

S. Eqn in Ket notation

$$\hat{H}\Psi(x,t) = i\hbar \frac{\partial}{\partial t} \Psi(x,t)$$

S. Eqn in wavefn notation.

$$\Psi(x,t) = \phi(x) T(t) \quad [\text{Assume separable}]$$

Coupled differential eqns

$$\left\{ \begin{array}{l} \hat{H} \phi_{E_n}(x) = E_n \phi_{E_n}(x) \\ \frac{dT}{dt} = -i \frac{E}{\hbar} T(t) \end{array} \right.$$

The eigenvalue eqn.

$$\Rightarrow T(t) = e^{-i \frac{E_n}{\hbar} t}$$

The general soln will be

$$\Psi(x,t) = \sum_n c_n e^{-i \frac{E_n}{\hbar} t} \phi_{E_n}(x)$$

Time dependence      space dependence.

## Exercise

Using mathematica, animate this wave state for a particle in an infinite potential well

$$|\Psi\rangle = \frac{1}{\sqrt{2}} |\phi_1\rangle + \frac{1}{\sqrt{2}} |\phi_2\rangle$$