DAY 12

Ph424

Consider the following wave state which describes an electron in an infinite potential well.

$$|\Phi\rangle = \sqrt{\tfrac{1}{5}}\,|\phi_1\rangle + \sqrt{\tfrac{3}{5}}\,|\phi_2\rangle + \sqrt{\tfrac{1}{5}}\,|\phi_3\rangle$$

which can be written in wave fn notation as

$$\Phi(x) \overset{@t=0}{=} \begin{cases} 0 & \text{I} \\ \sqrt{\tfrac{1}{5}}\sqrt{\tfrac{2}{L}}\sin\frac{\pi x}{L} + \sqrt{\tfrac{3}{5}}\sqrt{\tfrac{2}{L}}\sin\frac{2\pi x}{L} + \sqrt{\tfrac{1}{5}}\sqrt{\tfrac{2}{L}}\sin\frac{3\pi x}{L} & \text{II} \\ 0 & \text{III} \end{cases}$$

Is this an energy eigenstate?

Answer: No, it does satisfy the eigen value eqn

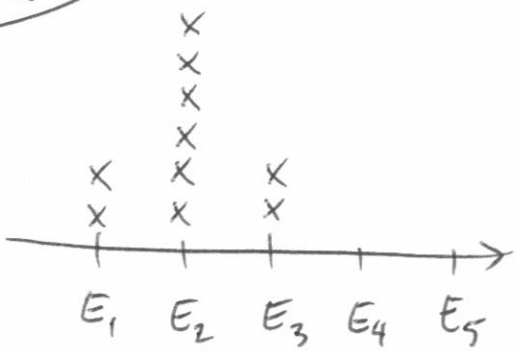
$$H|\Phi\rangle \neq \text{const}\,|\Phi\rangle$$

What are the possible energys I could measure?

Answer: E_1 E_2 & E_3

If I make lots of measurements in of this state (always preparing the state exactly the same way) what will be the distribution of results?

Answer



$$\mathcal{P}_{E_1} = 1/5$$

$$\mathcal{P}_{E_2} = 3/5$$

$$\mathcal{P}_{E_3} = 1/5$$

What is $\langle E \rangle$? i.e. Calculate $\langle E \rangle$.

Answer $\langle \Phi | H | \Phi \rangle =$

$$= \langle \Phi | H \left(\sqrt{\tfrac{1}{5}} |\phi_1\rangle + \sqrt{\tfrac{3}{5}} |\phi_2\rangle + \sqrt{\tfrac{1}{5}} |\phi_3\rangle \right)$$

$$= \langle \Phi | \left(\sqrt{\tfrac{1}{5}} E_1 |\phi_1\rangle + \sqrt{\tfrac{3}{5}} E_2 |\phi_2\rangle + \sqrt{\tfrac{1}{5}} E_3 |\phi_3\rangle \right)$$

$$= \frac{1}{5} E_1 + \frac{3}{5} E_2 + \frac{1}{5} E_3$$

Calculate ΔE

Answer $\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$

$$\langle E^2 \rangle = \langle \Phi | \hat{H} \hat{H} | \Phi \rangle$$

$$= \frac{1}{5} E_1^2 + \frac{3}{5} E_2^2 + \frac{1}{5} E_3^2$$

Then work thru the algebra to find ΔE.

Note: These calculation can also be done as wave function integrals. However, ket notation is typically more convenient. There are some calculations where you have no choice but to use wavefn integrals.

Example: Find the expectation value of position

$$\langle \hat{x} \rangle = \langle \Phi | \hat{x} | \Phi \rangle$$

The operator used to measure position

$$= \int_{-\infty}^{\infty} \Phi^*(x)\, x\, \Phi(x)\, dx$$

Example: Find the expectation value of momentum

$$\langle \hat{p} \rangle = \langle \Phi | -i\hbar \frac{d}{dx} | \Phi \rangle$$

$$= \int_{-\infty}^{\infty} \Phi^*(x) \left(-i\hbar \frac{d}{dx}\right) \Phi(x)\, dx$$

$$= \int_0^L \Phi^*(x)\left(-i\hbar\frac{d}{dx}\right)\left(\sqrt{\frac{1}{5}}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} + \sqrt{\frac{3}{5}}\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L} + \sqrt{\frac{1}{5}}\sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}\right)dx$$

You will end up with 9 cross terms

$$\int_0^L \sin\frac{\pi x}{L}\cos\frac{\pi x}{L}\, dx$$

$$\int_0^L \sin\frac{\pi x}{L}\cos\frac{2\pi x}{L}\, dx \qquad \text{etc.}$$

Everything is multiplied by i

$\langle \hat{p} \rangle$ cannot be imaginary, so the integrals must add up to zero.

For a physical argument: We know the electron is ~~can~~ not escaping to the left or right.

$\Rightarrow \langle \hat{p} \rangle = 0.$

NOW ADD TIME DEPENDENCE

$$\hat{H}|\Psi\rangle = i\hbar\frac{\partial}{\partial t}|\Psi\rangle$$ S. Eqn in Ket notation

$$\hat{H}\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$ S. Eqn in wavefn notation.

$$\Psi(x,t) = \phi(x)\,T(t)$$ [Assume separable]

Coupled differential eqns

$$\left\{\begin{array}{l} \hat{H}\phi_{E_n}(x) = E_n\phi_{E_n}(x) \\ \dfrac{dT}{dt} = -i\dfrac{E}{\hbar}T(t) \end{array}\right.$$

The eigenvalue eqn.

$$\Rightarrow T(t) = e^{-i\frac{E_n}{\hbar}t}$$

The general soln will be

$$\Psi(x,t) = \sum_n c_n \underbrace{e^{-i\frac{E_n}{\hbar}t}}_{\text{Time dependence}} \underbrace{\phi_{E_n}(x)}_{\text{space dependence.}}$$

Exercise: Using mathematica, animate this wave state for a particle in an infinite potential well

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$$