(*Hint:* It is *not* a good approximation to assume that this range of speeds is small enough that we can just multiply $\mathcal{D}(v)$ by $dv = 0.1v_P$. Why not? So you will have to do an integral over $\mathcal{D}(v)$. But you can replace the exponential by something simpler over this range. You can *check* your work with MBoltz.)

T6M.4 In this problem, we will consider the energy budget required for a human body to emit thermal photons.

(a) Consider a relatively small person with a surface area of $1.0\ \text{m}^2$ and a body temperature of 37°C = 310 K. Show that such a person would need to consume about 11,000 food calories a day simply to replenish the energy emitted by his or her body. (Assume that a person's emissivity $\epsilon \approx 1$ in the relevant photon energy range.)

(b) Of course, this is for a person who is naked in a vacuum. A small person actually requires something more like 2000 food calories a day to maintain body temperature. This is partly because a person's typical surroundings emit thermal photons back at the person. Calculate the *net* energy per day (in food calories) that a naked person in a room at 295 K would emit. (*Hint:* Equation T6.25 also describes the power that a surface *absorbs* from surroundings at absolute temperature T.)

(c) Clothing also helps. Suppose that the person's entire surface area A is surrounded by a layer of clothing that completely absorbs the person's thermal photons, but also emits photons in both directions from its surface, half back to the person from the layer's inner surface and half to the surroundings from its outer surface. Let P_p/A be the power per unit area emitted by the person, let P_c/A be the power per unit area emitted by *each surface* of the clothing layer, and let P_s/A be the power per unit area that the layer absorbs from the room. When the clothing layer is in equilibrium with both the person at 310 K and its surroundings at 295 K, then the net energy flow to that layer must be zero, meaning that $P_p/A + P_s/A = 2P_c/A$ (since the layer has *two* surfaces of area A that are both emitting). Show that the layer must have absolute temperature $T_c = 302.8$ K.

(d) Calculate, therefore, the *net* rate at which a clothed person loses energy in food calories per day.

T6M.5 Use some kind of computer tool to construct plots of $(1/A)(dP(\varepsilon)/d\varepsilon)$ (the blackbody's radiated power per unit area per unit energy range) as a function of photon energy ε (in eV) for photon gases at 1000 K, 1500 K, and 2000 K. Draw all three plots on the same graph, and comment about how the three graphs compare.

T6M.6 Suppose that we paint a certain object with red paint so that it absorbs all light falling on it *except* for red light in a photon energy range between 1.8 eV and 2.0 eV, which it completely reflects. Suppose now that this object has a temperature such that $\varepsilon_P = 1.40$ eV. Sketch a graph of $dP(\varepsilon)/P_b$ for the energy emitted by this object (don't worry about the numbers on the vertical axis), and carefully explain why it *cannot* look the same as figure T6.4 if the object is to remain in thermal equilibrium in all cases.

T6M.7 Suppose I want to skew the color of the photon gas in a container, so I go inside the container and quickly paint its inside walls with green paint that *almost* completely reflects green light corresponding to photon energies between 2.4 eV and 2.5 eV. Will this change the energy distribution of photons in the photon gas? If so, will the gas become less green or more green, and why? If not, why not?

Derivation

T6D.1 The total probability of a molecule having a speed between zero and infinity must be 1. By looking up the integral online (or using WolframAlpha), evaluate the constant A in the expression

$$1 = \int_0^\infty \mathcal{D}(v)\frac{dv}{v_P} = \int_0^\infty A\left(\frac{v}{v_P}\right)^2 e^{-(v/v_P)^2}\left(\frac{dv}{v_P}\right) \qquad \text{(T6.28)}$$

(*Hint:* Define $x = v/v_P$ before looking up the integral.)

T6D.2 Prove that the maximum of the Maxwell-Boltzmann distribution function $\mathcal{D}(v)$ is precisely at $v = v_P$.

T6D.3 The equation below (equation T6.8) gives the average value of a molecule's speed in a gas:

$$v_{\text{avg}} = \int_0^\infty v\left[\mathcal{D}(v)\frac{dv}{v_P}\right] = v_P\int_0^\infty \left(\frac{v}{v_P}\right)\mathcal{D}(v)\frac{v}{v_P} \qquad \text{(T6.29)}$$

By looking up the integral online (or using WolframAlpha, show that $v_{\text{avg}} = \sqrt{(4/\pi)}\ v_P = \sqrt{8k_BT/\pi m}$ (equation T6.9). (*Hint:* Substitute in $x = v/v_P$ before looking up the integral.)

T6D.4 The equation below gives the average value of the square of a molecule's speed:

$$[v^2]_{\text{avg}} = \int_0^\infty v^2\left[\mathcal{D}(v)\frac{dv}{v_P}\right] = v_P^2\int_0^\infty \left(\frac{v}{v_P}\right)^2\mathcal{D}(v)\frac{v}{v_P} \qquad \text{(T6.30)}$$

(a) By looking up the integral online (or using WolframAlpha), find the value of $[v^2]_{\text{avg}}$. (*Hint:* Substitute in $x = v/v_P$ before looking up the integral.)

(b) Explain why your answer makes sense. (*Hint:* What do we know about K_{avg}?)

T6D.5 In this problem, we will calculate the number of photons in a photon gas at absolute temperature T.

(a) The number of photons whose energies lie within the range $\pm\frac{1}{2}d\varepsilon$ about some energy ε is equal to the total photon energy in that range divided by ε, right? Use this to set up the integral that you need to calculate the number of photons per unit volume N/V in the gas.

(b) If you change variables to integrate over the unitless variable $u = \varepsilon/k_BT$, you should see that N/V depends on $(k_BT)^3$. Explain why knowing the specific value of the integral over u is irrelevant for this claim.

(c) By looking up the integral (perhaps on Wikipedia's list of integrals), show that the number of photons per unit volume in a photon gas with temperature T is

$$\frac{N}{V} = 8\pi\cdot 2.404\left(\frac{k_BT}{hc}\right)^3 = 60.42\left(\frac{k_BT}{hc}\right)^3 \qquad \text{(T6.31)}$$