Uncertainties and error analysis

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error The difference between your result and the exact result.

uncertainty An estimate of how big you think your error is likely to be.

Propagating uncertainties

Propagating uncertainties is the process of deciding the uncertainty of a computed quantity based on the uncertainties of measured quantities. It does require you to either measure your uncertainties in your measurements (which is possible, but you don't have the data to accomplish for this lab) or to estimate those uncertainties. When estimating an uncertainty, you always want to estimate how big you think it is, rather than trying to pick a number so large that the uncertainty must surely be less than that. If your error turns out way smaller than your uncertainty, then either you were lucky (which is unlikely) or you did a poor job of estimating uncertainties.

Now let's assume that you know the uncertainty of your measurements. How do you convert that to an uncertainty of your prediction, i.e. your final result?

Area of a square

> **Small groups** *Suppose you measure the side of a square to be* $\ell = 5.0$ *cm with an uncertainty of* $\delta\ell \approx 0.1$ *cm. Try as a group to estimate the uncertainty in the area of the square.*

There are a couple of approaches you might have tried. One would be to consider the extreme cases:

$$\begin{align} \delta A &= \frac{A(\ell = 5.1\text{ cm}) - A(\ell = 4.9\text{ cm})}{2} \tag{1}\\ &= \tfrac{1}{2}(5.1^2 - 4.9^2)\text{ cm}^2 \tag{2}\\ &= 1\text{ cm}^2 \tag{3} \end{align}$$

Note that I divided by 2 to get a "±" uncertainty rather than a "full width" uncertainty.

You might instead have recognized that we're multiplying $\ell \cdot \ell$, and thought to combine errors in a Pythagorean way, as we do when multiplying:

$$\frac{\delta A}{A} = \sqrt{\left(\frac{0.1}{5}\right)^2 + \left(\frac{0.1}{5}\right)^2} \tag{4}$$

$$= 0.02\sqrt{2} \tag{5}$$

$$\delta A = 0.02(25 \text{ cm}^2)\sqrt{2} \tag{6}$$

$$= 0.7 \text{ cm}^2 \tag{7}$$

Which of these solutions is correct? They differ by a factor of $\sqrt{2}$. The answer is that the first one is correct. The Pythagorean way to combine uncertainties assumed that the uncertainties were uncorrelated, but in this case, when the first ℓ is bigger than 5 cm, then the second one is also bigger.

If we have just one measurement as your input (and your errors are reasonably small), we can propagate our uncertainty using a derivative. If you know the value of x with an uncertainty of δx, then your uncertainty of $f(x)$ is given by

$$\delta f = \frac{df}{dx}\delta x \tag{8}$$

You can look at the dimensions of this expression to convince yourself that it is reasonable, or you can look at Fig. 1.

Measuring two sides

Suppose now that we separately measure both sides of the square (w and h), find each side to have length 5.0 cm, and compute the area via $A = wh$. If the uncertainty of both measurements are equal to 0.1 cm (the same as last time), how do we find our uncertainty in the area? The answer should be different than when we made just one measurement.

> **Small white boards** *Do you expect the uncertainty to be greater or less than the case where we made only one measurement?*

The uncertainty should be less, because although both errors are the same size (on average) as in our previous case, their effects may cancel out, if one error is positive and the other error is negative. To take this into account, we need to combine our uncertainties using a Pythagorean approach

$$\boxed{\delta f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \delta x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \delta y^2 + \cdots}} \tag{9}$$

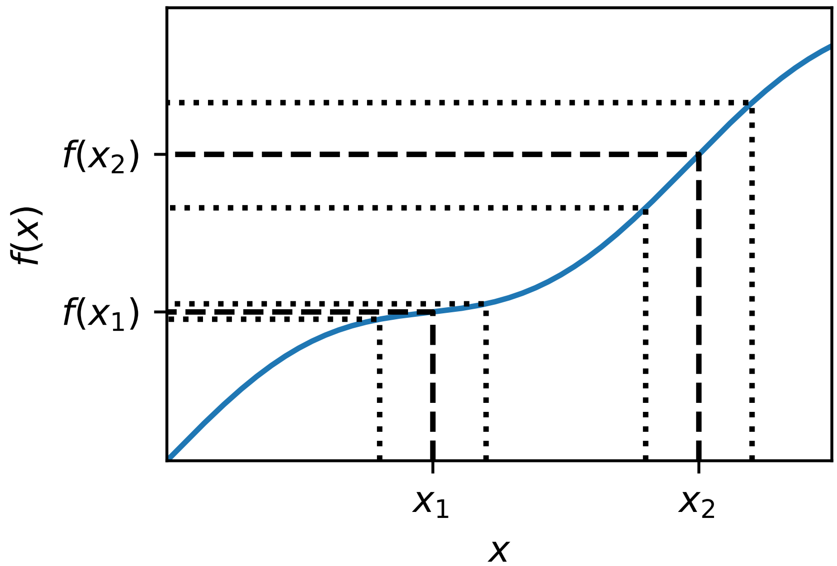

Figure 1: Graphical picture of why the uncertainty of $f(x)$ depends on its slope.

For the area of the square, this becomes

$$\delta A = \sqrt{\left(\frac{\partial A}{\partial w}\right)^2 \delta w^2 + \left(\frac{\partial A}{\partial h}\right)^2 \delta h^2} \tag{10}$$

$$= \sqrt{h^2 \delta w^2 + w^2 \delta h^2} \tag{11}$$

$$= \sqrt{(5.0 \text{ cm})^2 (0.1 \text{ cm})^2 + (5.0 \text{ cm})^2 (0.1 \text{ cm})^2} \tag{12}$$

$$= \frac{1 \text{ cm}^2}{\sqrt{2}} \tag{13}$$

So the uncertainty is $\sqrt{2}$ times smaller when we made two measurements than when we made just one.

In general

Equation 9 in the box above tells you how to propagate uncertainties. The approaches you've been taught in this class are equivalent to this equation, whereas in earlier classes you may have been taught crude approximations of this equation.[1] In order to determine your final uncertainty you need to have a symbolic expression for the final answer in terms of your measurements.

[1] I will mention one caveat: this expression does assume that your uncertainties are small enough that nonlinear effects can be neglected. Usually our certainty of our uncertainties is

Small groups *Work out an expression for the uncertainty of the quantity your group is assigned in terms of its component parts. In each case, try to express your uncertainty in the prettiest way. This may either be as a fractional error ($\delta A/A$) or as an absolute error. In each case assume there is some known uncertainty on each variable on the right hand side of the expression.*

1. *The area of an ellipse:*

$$A = \pi ab$$

2. *The perimeter of a rectangle:*

$$p = 2a + 2b$$

3. *The Carnot efficiency:*

$$\eta = 1 - \frac{T_c}{T_H}$$

4. *The wavelength of light:*

$$\lambda = d \sin\theta$$

5. *Finding the wavelength from the distances of two legs of a right triangle:*

$$\lambda = d\frac{y}{\sqrt{x^2+y^2}} \tag{14}$$

Answer to #5 above

$$\delta\lambda = \sqrt{\left(\frac{\partial\lambda}{\partial d}\right)^2 \delta d^2 + \left(\frac{\partial\lambda}{\partial x}\right)^2 \delta x^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 \delta y^2} \tag{15}$$

$$= \sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 \delta d^2 + d^2\left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{2}\frac{y}{(x^2+y^2)^{\frac{3}{2}}}2y\right)^2 \delta x^2 + \left(-d\frac{1}{2}\frac{y}{(x^2+y^2)^{\frac{3}{2}}}2x\right)^2 \delta y^2} \tag{16}$$

$$= \sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 \delta d^2 + d^2\left(\frac{1}{\sqrt{x^2+y^2}}\frac{x^2+y^2}{x^2+y^2} - \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\right)^2 \delta x^2 + \left(d\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\right)^2 \delta y^2} \tag{17}$$

$$= \sqrt{\frac{y^2}{x^2+y^2}\delta d^2 + \frac{d^2x^4}{(x^2+y^2)^3}\delta x^2 + \frac{d^2x^2y^2}{(x^2+y^2)^3}\delta y^2} \tag{18}$$

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low enough that this is an unimportant caveat, but in high energy physics it is common to instead use a numerical simulation to propagate uncertainties.

It's kind of a mess. You can maybe clean it up a bit by putting λ back in, and you get:

$$\frac{\delta\lambda}{\lambda} = \sqrt{\frac{\delta d^2}{d^2} + \frac{x^4}{y^2(x^2+y^2)^2}\delta x^2 + \frac{x^2}{(x^2+y^2)^2}\delta y^2} \tag{19}$$

Sources of error

> **Small groups** *Write down on index cards all the possible sources of error that you can think of in the LED experiment you just did.*

mistake Something you did that was not what you should have done when taking a measurement or analyzing your results.

random error A source of error that will be different each time you take a measurement, such that an average of many measurements will approach zero error.

systematic error A **systematic** error is a source of error that does not change with each measurement.

> **Small groups** *Categorize your sources of error as random, systematic, or mistakes.*

Random

1. Inaccuracy in measuring distances (e.g. rounding to nearest millimeter).
2. Error due to the width of the beam in terms of angles, which makes it hard to pick the "center" of the beam.
3. Limited number of digits on the multimeter, or perhaps random fluctuation in the multimeter readings.
4. Maybe the beam wasn't aligned at an angle of zero, but each time you put in a new LED it had a random alignment error.

Systematic

1. Maybe the diffraction grating doesn't have exactly a 1μm spacing?
2. Maybe your ruler didn't quite reach the grating, adding a constant value each time.
3. What if there was a problem in the IV extrapolation method you used, which could give you a voltage that is too high or too low (and consistently).
4. The LED produces a range of photon energies (if the current is high enough), leading to a range of colors. Perhaps the photons you measured did not actually have the energy you measured.

Mistakes

1. Did you round the electron charge to two significant figures?
2. Getting mixed up which data was for which LED.
3. Maybe you measured distances from the wrong point entirely?
4. Maybe you din't use extrapolation, but instead just measured the voltage for a single current?

Sometimes it can be hard to distinguish between random and systematic errors. If you don't repeat a measurement then in a sense all errors are systematic.

The great thing about random errors is that you can measure your uncertainty! You can just repeat a measurement a few times, and see how close the answers come. You can make this quantitative by computing the standard deviation of your measurements. On top of this, you can reduce your random error by repeating your measurement and then taking an average. So yay for random errors!

Systematic errors are terrifying. There is no way to measure how big they are. There is no way to fix them, except by identifying and understanding them. The solution is to try to imagine any possible systematic error, and then design your experiment such that the systematic errors are as small as possible, or even better, won't affect your final answer.