15.1.1 We want to find the $\mathbf{S}^2$ operator in the uncoupled basis $\left|m_1 m_2\right\rangle$, which comprises four states:

$$\left|++\right\rangle ,\ \left|+-\right\rangle ,\ \left|-+\right\rangle ,\ \left|--\right\rangle$$

The $\mathbf{S}^2$ operator is

$$\mathbf{S}^2 = \left(\mathbf{S}_1 + \mathbf{S}_2\right)^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \bullet \mathbf{S}_2$$

Let's do each piece in turn. The eigenvalue equations for $\mathbf{S}_1^2$ and $\mathbf{S}_2^2$ are

$$\mathbf{S}_1^2\left|m_1 m_2\right\rangle = s_1\left(s_1+1\right)\hbar^2\left|m_1 m_2\right\rangle$$
$$\mathbf{S}_2^2\left|m_1 m_2\right\rangle = s_2\left(s_2+1\right)\hbar^2\left|m_1 m_2\right\rangle$$

where $s_1 = 1/2$ and $s_2 = 1/2$. Now use these to find the matrix elements:

$$\left\langle m_1' m_2'\right|\mathbf{S}_1^2\left|m_1 m_2\right\rangle = \left\langle m_1' m_2'\right|s_1\left(s_1+1\right)\hbar^2\left|m_1 m_2\right\rangle = s_1\left(s_1+1\right)\hbar^2\left\langle m_1' m_2' \middle| m_1 m_2\right\rangle = s_1\left(s_1+1\right)\hbar^2\delta_{m_1 m_1'}\delta_{m_2 m_2'}$$
$$\left\langle m_1' m_2'\right|\mathbf{S}_2^2\left|m_1 m_2\right\rangle = \left\langle m_1' m_2'\right|s_2\left(s_2+1\right)\hbar^2\left|m_1 m_2\right\rangle = s_2\left(s_2+1\right)\hbar^2\left\langle m_1' m_2' \middle| m_1 m_2\right\rangle = s_2\left(s_2+1\right)\hbar^2\delta_{m_1 m_1'}\delta_{m_2 m_2'}$$

yielding

$$\mathbf{S}_1^2 \doteq \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{matrix} ++ \\ +- \\ -+ \\ -- \end{matrix}$$

$$\mathbf{S}_2^2 \doteq \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{matrix} ++ \\ +- \\ -+ \\ -- \end{matrix}$$

So each is proportional to the identity matrix.
Now work on the cross term:

$$\mathbf{S}_1 \bullet \mathbf{S}_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$$

Rewrite this in terms of the ladder operators, which are

$$S_{1+} = S_{1x} + iS_{1y} \quad S_{2+} = S_{2x} + iS_{2y}$$
$$S_{1-} = S_{1x} - iS_{1y} \quad S_{2-} = S_{2x} - iS_{2y}$$

Solve these for the Cartesian components:

$$S_{1x} = \tfrac{1}{2}\left(S_{1+} + S_{1-}\right) \quad S_{2x} = \tfrac{1}{2}\left(S_{2+} + S_{2-}\right)$$
$$S_{1y} = \tfrac{-i}{2}\left(S_{1+} - S_{1-}\right) \quad S_{2y} = \tfrac{-i}{2}\left(S_{2+} - S_{2-}\right)$$

and substitute to get

$$\begin{aligned}\mathbf{S}_1 \bullet \mathbf{S}_2 &= S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} \\ &= \tfrac{1}{2}\left(S_{1+}+S_{1-}\right)\tfrac{1}{2}\left(S_{2+}+S_{2-}\right) + \tfrac{-i}{2}\left(S_{1+}-S_{1-}\right)\tfrac{-i}{2}\left(S_{2+}-S_{2-}\right) + S_{1z}S_{2z} \\ &= \tfrac{1}{4}\left(S_{1+}S_{2+}+S_{1-}S_{2+}+S_{1+}S_{2-}+S_{1-}S_{2-}\right) - \tfrac{1}{4}\left(S_{1+}S_{2+}-S_{1-}S_{2+}-S_{1+}S_{2-}+S_{1-}S_{2-}\right) + S_{1z}S_{2z} \\ &= \tfrac{1}{2}\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right) + S_{1z}S_{2z}\end{aligned}$$

The ladder operators yield zero when acting on the extreme states

$$\begin{aligned}S_{1+}\left|++\right\rangle = S_{1+}\left|+-\right\rangle = S_{1-}\left|-+\right\rangle = S_{1-}\left|--\right\rangle = 0 \\ S_{2+}\left|++\right\rangle = S_{2+}\left|-+\right\rangle = S_{2-}\left|+-\right\rangle = S_{2-}\left|--\right\rangle = 0\end{aligned}$$

For the other states, use the ladder operator equation

$$J_\pm\left|j,m_j\right\rangle = \hbar\left[j(j+1)-m_j(m_j\pm 1)\right]^{1/2}\left|j,m_j\pm 1\right\rangle$$

which gives

$$\begin{aligned}S_{1+}\left|-+\right\rangle &= \hbar\left[s_1(s_1+1)-m_1(m_1+1)\right]^{1/2}\left|++\right\rangle = \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}\left|++\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|++\right\rangle \\ &= \hbar\left|++\right\rangle\end{aligned}$$

The other results are

$$\begin{aligned}S_{1+}\left|--\right\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}\left|+-\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|+-\right\rangle = \hbar\left|+-\right\rangle \\ S_{1-}\left|++\right\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}\left|-+\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|-+\right\rangle = \hbar\left|-+\right\rangle \\ S_{1-}\left|+-\right\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}\left|--\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|--\right\rangle = \hbar\left|--\right\rangle \\ S_{2+}\left|+-\right\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}\left|++\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|++\right\rangle = \hbar\left|++\right\rangle \\ S_{2+}\left|--\right\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}\left|-+\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|-+\right\rangle = \hbar\left|-+\right\rangle \\ S_{2-}\left|++\right\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}\left|+-\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|+-\right\rangle = \hbar\left|+-\right\rangle \\ S_{2-}\left|-+\right\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}\left|--\right\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}\left|--\right\rangle = \hbar\left|--\right\rangle\end{aligned}$$

The action of $\mathbf{S}_1 \bullet \mathbf{S}_2$ on the basis states $\left|m_1 m_2\right\rangle$ is

$$\begin{aligned}\mathbf{S}_1 \bullet \mathbf{S}_2\left|++\right\rangle &= \left\{\tfrac{1}{2}\left(S_{1+}S_{2-}+S_{1-}S_{2+}\right)+S_{1z}S_{2z}\right\}\left|++\right\rangle = \left\{\tfrac{1}{2}(0+0)+\tfrac{1}{2}\hbar\tfrac{1}{2}\hbar\right\}\left|++\right\rangle = \tfrac{1}{4}\hbar^2\left|++\right\rangle \\ \mathbf{S}_1 \bullet \mathbf{S}_2\left|--\right\rangle &= \left\{\tfrac{1}{2}(0+0)+\left(\tfrac{-1}{2}\right)\hbar\left(\tfrac{-1}{2}\right)\hbar\right\}\left|--\right\rangle = \tfrac{1}{4}\hbar^2\left|--\right\rangle \\ \mathbf{S}_1 \bullet \mathbf{S}_2\left|+-\right\rangle &= 0+\tfrac{1}{2}\hbar\hbar\left|-+\right\rangle+\tfrac{1}{2}\hbar\left(\tfrac{-1}{2}\right)\hbar\left|+-\right\rangle = \tfrac{1}{4}\hbar^2\left(2\left|-+\right\rangle-\left|+-\right\rangle\right) \\ \mathbf{S}_1 \bullet \mathbf{S}_2\left|-+\right\rangle &= 0+\tfrac{1}{2}\hbar\hbar\left|+-\right\rangle+\tfrac{1}{2}\hbar\left(\tfrac{-1}{2}\right)\hbar\left|-+\right\rangle = \tfrac{1}{4}\hbar^2\left(2\left|+-\right\rangle-\left|-+\right\rangle\right)\end{aligned}$$

Projecting these results onto the basis states yields the matrix representation

$$\mathbf{S}_1 \bullet \mathbf{S}_2 \doteq \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} ++ \\ +- \\ -+ \\ -- \end{matrix}$$

Now add the three parts to get

$$\begin{aligned} \mathbf{S}^2 &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \bullet \mathbf{S}_2 \\ &\doteq 2\frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 2\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\doteq \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{matrix} ++ \\ +- \\ -+ \\ -- \end{matrix} \end{aligned}$$

This operator is block diagonal, so we know two eigenvalues and eigenstates by inspection (we know that the eigenvalues have the form $s(s+1)\hbar^2$):

$$s_a = 1, \qquad |s_a = 1, m_a = 1\rangle = |++\rangle$$
$$s_b = 1, \qquad |s_b = 1, m_b = -1\rangle = |--\rangle$$

The other two eigenvalues and eigenstates are found by diagonalizing the submatrix in the middle

$$\begin{vmatrix} \hbar^2 - \lambda & \hbar^2 \\ \hbar^2 & \hbar^2 - \lambda \end{vmatrix} = 0$$

$$\begin{aligned} &(\hbar^2 - \lambda)^2 - (\hbar^2)^2 = 0 \\ &(\hbar^2 - \lambda) = \pm(\hbar^2) \\ &\lambda = \hbar^2 \pm \hbar^2 = 2\hbar^2, 0\hbar^2 \\ &s = 1, 0 \end{aligned}$$

The resultant eigenstates are superpositions of the two states $|+-\rangle$ and $|-+\rangle$:

$$s_c = 1: \begin{pmatrix} \hbar^2 & \hbar^2 \\ \hbar^2 & \hbar^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 2\hbar^2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \alpha + \beta = 2\alpha \Rightarrow \alpha = \beta \Rightarrow |s_c\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}$$

$$s_d = 0: \begin{pmatrix} \hbar^2 & \hbar^2 \\ \hbar^2 & \hbar^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0\hbar^2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \alpha + \beta = 0 \Rightarrow \alpha = -\beta \Rightarrow |s_d\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

15.1.2 (1) The hyperfine Hamiltonian is

$$H_{hf} = A\mathbf{S}_1 \bullet \mathbf{S}_2$$

We know from problem 15.1.1 that $\mathbf{S}_1 \bullet \mathbf{S}_2$ is nondiagonal when expressed in the uncoupled basis. However, it is diagonal in the coupled basis. This is clear if we note that

$$\mathbf{S}^2 = \left(\mathbf{S}_1 + \mathbf{S}_2\right)^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \bullet \mathbf{S}_2$$

$$\Rightarrow \mathbf{S}_1 \bullet \mathbf{S}_2 = \tfrac{1}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right)$$

The coupled basis vectors $\left|sm; s_1 s_2\right\rangle \equiv \left|sm\right\rangle$ are eigenstates of $\mathbf{S}^2$, S_z, $\mathbf{S}_1^2$, and $\mathbf{S}_2^2$. All the coupled states have the same quantum numbers $s_1 = 1/2$ and $s_2 = 1/2$, and hence are eigenstates of $\mathbf{S}_1^2$ and $\mathbf{S}_2^2$ with eigenvalues $s_i\left(s_i + 1\right)\hbar^2 = 3\hbar^2/4$. The matrices are thus proportional to the identity matrix (as they are in the uncoupled basis)

$$\mathbf{S}_1^2 \doteq \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 11 \\ 10 \\ 1,-1 \\ 00 \end{matrix}$$

$$\mathbf{S}_1^2 \doteq \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 11 \\ 10 \\ 1,-1 \\ 00 \end{matrix}$$

where the rows (and columns) are labeled with the s,m quantum numbers. The matrix for $\mathbf{S}^2$ is obtained from the eigenvalue equation $\mathbf{S}^2\left|sm\right\rangle = s\left(s+1\right)\hbar^2\left|sm\right\rangle$:

$$\mathbf{S}^2 \doteq \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} 11 \\ 10 \\ 1,-1 \\ 00 \end{matrix}$$

The hyperfine Hamiltonian is thus

$$H_{hf} = \frac{1}{2}A\left\{\hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

$$= \frac{A\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{matrix} 11 \\ 10 \\ 1,-1 \\ 00 \end{matrix}$$

Hence we can read the hyperfine energies from the diagonal values. These add to the values for the original Hamiltonian, giving

$$E = \begin{cases} -Ry + A\hbar^2/4; & s = 1 \\ -Ry - 3A\hbar^2/4; & s = 0 \end{cases}$$

There are 3 $s = 1$ states ($|11\rangle, |10\rangle, |1,-1\rangle$) and 1 $s = 0$ state ($|00\rangle$), the triplet and singlet.
(2) The energy difference between these 2 levels is

$$\Delta E = E(s=1) - E(s=0) = A\hbar^2 \equiv hf_{hf}$$

To estimate the frequency f_{hf} of this hyperfine (*hf*) transition, note that the interaction energy of the two magnetic dipoles separated by a distance a_0 is

$$E_{hf} \approx \frac{\vec{\boldsymbol{\mu}}_e \bullet \vec{\boldsymbol{\mu}}_P}{a_0^3}$$

where the magnetic moments are given by Eqn. 14.4.18b:

$$\vec{\boldsymbol{\mu}} = g\frac{q}{2mc}\vec{\mathbf{S}}$$

Hence we get

$$A\mathbf{S}_1 \bullet \mathbf{S}_2 \approx \frac{1}{a_0^3} g_e \frac{q}{2m_e c}\vec{\mathbf{S}}_1 \bullet g_p \frac{q}{2m_p c}\vec{\mathbf{S}}_2$$

$$\Rightarrow A \approx \frac{1}{a_0^3}\frac{eg_e}{2m_e c}\frac{eg_p}{2m_p c}$$

$$\approx \frac{1}{a_0^3}\frac{e(2)}{2m_e c}\frac{e(5.6)}{2m_p c}$$

Hence the energy difference is

$$\Delta E = A\hbar^2$$

$$\approx \frac{1}{a_0^3}\frac{e(2)}{2m_e c}\frac{e(5.6)}{2m_p c}\hbar^2$$

$$\approx \left(\frac{m_e e^2}{\hbar^2}\right)^3 \frac{e(2)}{2m_e c}\frac{e(5.6)}{2m_p c}\hbar^2 \approx 2.8\frac{e^8}{\hbar^4 c^4}\frac{m_e}{m_p}m_e c^2$$

$$\approx (2.8)\alpha^2\frac{m_e}{m_p}\alpha^2 m_e c^2 \approx (5.6)\alpha^2\frac{m_e}{m_p}Ryd$$

This gives a value of

$$\Delta E \approx (5.6)\frac{1}{137^2}\frac{1}{1836}13.6eV \approx 2.21\mu eV$$

and a wavelength of

$$\lambda \approx \frac{1240eVnm}{\Delta E} \approx \frac{1240eVnm}{2.21\mu eV} \approx 56cm$$

compared to the actual value of 21 cm. Our estimate for the frequency is

$$f_{hf} \approx \frac{c}{\lambda} \approx \frac{3\times 10^{10}\, cms^{-1}}{56cm} \approx 534\, MHz$$

compared to the actual value of 1420 MHz.

(3) To estimate the thermal populations use the Boltzmann factor (note the degeneracy factor):

$$\frac{P_{s=1}}{P_{s=0}} = \frac{g_{s=1}\ e^{-E_{s=1}/kT}}{g_{s=0}\ e^{-E_{s=0}/kT}} = 3e^{-\left(E_{s=1}-E_{s=0}\right)/kT} = 3e^{-\Delta E/kT}$$

$$\approx 3e^{-2.21\mu eV/25meV}$$

$$\approx 3\left(1-\frac{2.21\mu eV}{25meV}\right) \approx 3\left(1-0.00009\right) \approx 3\times 0.99991 \approx 2.99973$$

3. Particle #1 has angular momentum 1 (j_1 = 1) and particle #2 has angular momentum 1/2 (j_2 = 1/2).

a) The possible uncoupled basis states $\left|j_1 m_1 j_2 m_2\right\rangle$ are:

There are 3 states with $j_1 = 1$, each with a different z-projection: $m_1 = 1,0,-1$

There are 2 states with $j_2 = 1/2$, each with a different z-projection $m_2 = 1/2, -1/2$.

There are 6 possible states in the uncoupled basis states $\left|j_1 m_1 j_2 m_2\right\rangle$. These are

$$\boxed{\begin{array}{cc} \left|11\tfrac{1}{2}\tfrac{1}{2}\right\rangle & \left|11\tfrac{1}{2}\tfrac{-1}{2}\right\rangle \\ \left|10\tfrac{1}{2}\tfrac{1}{2}\right\rangle & \left|10\tfrac{1}{2}\tfrac{-1}{2}\right\rangle \\ \left|1,-1\tfrac{1}{2}\tfrac{1}{2}\right\rangle & \left|1,-1\tfrac{1}{2}\tfrac{-1}{2}\right\rangle \end{array}}$$

b) For any angular momentum addition, the possible values are $J = j_1 + j_2\ ,\ j_1 + j_2 - 1\ ,\ j_1 + j_2 - 2\ ,\ \ldots \left|j_1 - j_2\right|$. In this case, we get

$$\boxed{J = \tfrac{3}{2}\ ,\ \tfrac{1}{2}}$$

The allowed values of M are always $-J$ to J, giving

$$\boxed{\begin{array}{l} J = \tfrac{3}{2}: \quad M = \tfrac{3}{2}\ ,\ \tfrac{1}{2}\ ,\ \tfrac{-1}{2}\ ,\ \tfrac{-3}{2} \\ J = \tfrac{1}{2}: \quad M = \tfrac{1}{2}\ ,\ \tfrac{-1}{2} \end{array}}$$

c) The coupled basis states are

$$\boxed{\begin{array}{llll} \left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle & \left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle & \left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle & \left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle \\ \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle & \left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle & & \end{array}}$$

d) The Clebsch-Gordan table is given below

$j_1=1$	$j_2=\frac{1}{2}$	j	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		m	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
m1	m2							
1	$\frac{1}{2}$		1	0	0	0	0	0
1	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{3}}$	0	0	$\sqrt{\frac{2}{3}}$	0
0	$\frac{1}{2}$		0	$\sqrt{\frac{2}{3}}$	0	0	$-\frac{1}{\sqrt{3}}$	0
0	$-\frac{1}{2}$		0	0	$\sqrt{\frac{2}{3}}$	0	0	$\frac{1}{\sqrt{3}}$
−1	$\frac{1}{2}$		0	0	$\frac{1}{\sqrt{3}}$	0	0	$-\sqrt{\frac{2}{3}}$
−1	$-\frac{1}{2}$		0	0	0	1	0	0

Using the columns of the Clebsch-Gordan table gives the coupled basis states in terms of the uncoupled basis states

$$\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \left|11\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|11\tfrac{1}{2}\tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|10\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|10\tfrac{1}{2}\tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1,-1\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle = \left|1,-1\tfrac{1}{2}\tfrac{-1}{2}\right\rangle$$
$$\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|11\tfrac{1}{2}\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|10\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|10\tfrac{1}{2}\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1,-1\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$

e) Using the rows of the Clebsch-Gordan table gives the uncoupled basis states in terms of the coupled basis states

$$\left|11\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle$$
$$\left|11\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$
$$\left|10\tfrac{1}{2}\tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$
$$\left|10\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle$$
$$\left|1,-1\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle$$
$$\left|1,-1\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle$$