12.3.4 The eigenstates of L_z are

$$\left|m\right\rangle \doteq \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

It is useful to write the wave function in terms of these eigenstates, giving

$$\begin{aligned}
\psi(\rho,\phi) &= Ae^{-\rho^2/2\Delta^2}\left(\frac{\rho}{\Delta}\cos\phi + \sin\phi\right) \\
&= Ae^{-\rho^2/2\Delta^2}\left(\frac{\rho}{\Delta}\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{e^{i\phi}-e^{-i\phi}}{2i}\right) \\
&= \frac{A}{2}e^{-\rho^2/2\Delta^2}\left[e^{i\phi}\left(\frac{\rho}{\Delta} - i\right) + e^{-i\phi}\left(\frac{\rho}{\Delta} + i\right)\right]
\end{aligned}$$

To find the probability of measuring L_z we project the wave function onto the L_z eigenstate in question, square the amplitude, and then sum over all possible ways to obtain that probability. If the state was expanded in terms of discrete basis states $\left|nm\right\rangle$, where the eigenvalues n refer to the other commuting observable (e.g. H), then we would express this as

$$\mathcal{P}_{L_z=mh} = \sum_{n=1}^{\infty}\left|\left\langle nm|\psi\right\rangle\right|^2$$

By inspection, we see that only the m = 1 and m = -1 states have non-zero probability. Without knowing the n basis states, we can proceed in a general way. Let the radial basis states be $R_{nm}(\rho)$. Write the wave function above in terms of two new radial functions

$$\begin{aligned}
\psi(\rho,\phi) &= \frac{e^{i\phi}}{\sqrt{2\pi}}\left[f(\rho) - ig(\rho)\right] + \frac{e^{-i\phi}}{\sqrt{2\pi}}\left[f(\rho) + ig(\rho)\right] \\
f(\rho) &= \sqrt{2\pi}\,\frac{A}{2}e^{-\rho^2/2\Delta^2}\,\frac{\rho}{\Delta} \\
g(\rho) &= \sqrt{2\pi}\,\frac{A}{2}e^{-\rho^2/2\Delta^2}
\end{aligned}$$

and expand each of these radial functions in the $R_{nm}(\rho)$ basis

$$\begin{aligned}
f(\rho) &= \sum_{n,m} a_{nm}R_{nm}(\rho) \\
g(\rho) &= \sum_{n,m} b_{nm}R_{nm}(\rho)
\end{aligned}$$

These are all real functions, so the coefficients are real. The probability in integral form is

$$\mathcal{P}_{L_z=mh} = \sum_{n=1}^{\infty}\left|\int_0^{\infty}\int_0^{2\pi} R^*_{nm}(\rho)\Phi^*_m(\phi)\psi(\rho,\phi)\,d\phi\rho\,d\rho\right|^2$$

Note that the two integrals are inside the absolute value to find probability amplitudes (say c_{nm}) and the sum is outside to add up all the possible probabilities ($\sum_n |c_{nm}|^2$). For *m* = 1, the angular projection selects just the $m = 1$ term, leaving the radial part that goes with it:

$$\begin{aligned}
\mathcal{P}_{L_z=1\hbar} &= \sum_{n=1}^{\infty} \left| \int_0^{\infty} R_{n1}^*(\rho)\left[f(\rho) - ig(\rho) \right] \rho \, d\rho \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \int_0^{\infty} R_{n1}^*(\rho)\left[\sum_{r,s} a_{rs} R_{rs}(\rho) - i \sum_{r,s} b_{rs} R_{rs}(\rho) \right] \rho \, d\rho \right|^2 \\
&= \sum_{n=1}^{\infty} \left| a_{n1} - ib_{n1} \right|^2 \\
&= \sum_{n=1}^{\infty} a_{n1}^2 + b_{n1}^2
\end{aligned}$$

Note that *a* and *b* are real. For *m* = -1, we get

$$\begin{aligned}
\mathcal{P}_{L_z=-1\hbar} &= \sum_{n=1}^{\infty} \left| \int_0^{\infty} R_{n,-1}^*(\rho)\left[f(\rho) + ig(\rho) \right] \rho \, d\rho \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \int_0^{\infty} R_{n,-1}^*(\rho)\left[\sum_{r,s} a_{rs} R_{rs}(\rho) + i \sum_{r,s} b_{rs} R_{rs}(\rho) \right] \rho \, d\rho \right|^2 \\
&= \sum_{n=1}^{\infty} \left| a_{n,-1} + ib_{n,-1} \right|^2 \\
&= \sum_{n=1}^{\infty} a_{n,-1}^2 + b_{n,-1}^2
\end{aligned}$$

The two probabilities appear to be different because the *m* values differ. But the differential equation that determines the $R_{nm}(\rho)$ basis states (see Eq. 12.3.13) includes an m^2 term and so cannot differentiate between positive and negative values of *m*. Thus the two probabilities for *m* = 1 and *m* = -1 must be equal. Because these two probabilities must add to 1, they are each equal to ½.

$$\mathcal{P}_{L_z=+1\hbar} = \frac{1}{2}$$
$$\mathcal{P}_{L_z=-1\hbar} = \frac{1}{2}$$

We could also solve the problem by using the continuous radial coordinate basis $|\rho\rangle$ and integrating over all possible values of that eigenvalue (see Eqn. 12.5.38 for 3D example):

$$\begin{aligned}
\mathcal{P}_{L_z=m\hbar} &= \int_0^{\infty} \left| \langle \rho m | \psi \rangle \right|^2 \rho \, d\rho \\
&= \int_0^{\infty} \left| \int_0^{2\pi} \Phi_m^*(\phi) \psi(\rho,\phi) \, d\phi \right|^2 \rho \, d\rho
\end{aligned}$$

Note that the angular integral is inside the absolute value to find the radial probability amplitude density (say $c_m(\rho)$) and the radial integral is outside to add up all the possible probabilities ($\int |c_m(\rho)|^2 \rho\, d\rho$). For the wave function given above, this results in

$$\mathcal{P}_{L_z=mh} = \int_0^\infty \left| \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{-im\phi} \frac{A}{2} e^{-\rho^2/2\Delta^2} \left[e^{i\phi}\left(\frac{\rho}{\Delta} - i\right) + e^{-i\phi}\left(\frac{\rho}{\Delta} + i\right)\right] d\phi \right|^2 \rho\, d\rho$$

For m = 1, we get

$$\begin{aligned}\mathcal{P}_{L_z=mh} &= \sqrt{2\pi} \int_0^\infty \left| \frac{A}{2} e^{-\rho^2/2\Delta^2} \left(\frac{\rho}{\Delta} - i\right) \right|^2 \rho\, d\rho \\ &= \sqrt{2\pi} \frac{|A|^2}{4} \int_0^\infty e^{-\rho^2/\Delta^2} \left(\frac{\rho^2}{\Delta^2} + 1\right) \rho\, d\rho\end{aligned}$$

For m = -1, we get

$$\begin{aligned}\mathcal{P}_{L_z=-mh} &= \sqrt{2\pi} \int_0^\infty \left| \frac{A}{2} e^{-\rho^2/2\Delta^2} \left(\frac{\rho}{\Delta} + i\right) \right|^2 \rho\, d\rho \\ &= \sqrt{2\pi} \frac{|A|^2}{4} \int_0^\infty e^{-\rho^2/\Delta^2} \left(\frac{\rho^2}{\Delta^2} + 1\right) \rho\, d\rho\end{aligned}$$

No need to do the integrals. We see that the two probabilities are equal and we know from inspection of the wave function that there are no other possible values of L_z. Hence, these two probabilities must add to 1, so they are each ½.

12.3.6 There is no potential energy here, so the energy is all kinetic. The energy of a classical particle rotating in a circular path in the x,y plane with a radius a is

$$E = K = \tfrac{1}{2}\mu v^2 = \frac{p^2}{2\mu} = \frac{(pa)^2}{2\mu a^2} = \frac{\ell_z^2}{2\mu a^2}$$

Hence the quantum mechanical Hamiltonian is

$$H = \frac{L_z^2}{2I}$$

where $I = \mu a^2$ is the moment of inertia. The eigenvalue equation is

$$H|\psi\rangle = E|\psi\rangle$$

Writing this in the coordinate basis yields

$$\begin{aligned}-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}\psi(\rho,\phi) &= E\psi(\rho,\phi) \\ \frac{\partial^2}{\partial\phi^2}\psi(\rho,\phi) &= -\frac{2IE}{\hbar^2}\psi(\rho,\phi)\end{aligned}$$

The solutions to this differential equation are the complex exponentials

$$\psi(\rho,\phi) = NR(\rho)e^{\pm i\left(\frac{\sqrt{2IE}}{\hbar}\right)\phi}$$

where N is the normalization constant and $R(\rho)$ is an arbitrary radial function. Now impose the condition (Eq. 12.3.6)

$$\psi(\rho,0) = \psi(\rho,2\pi)$$

which requires that the factoring multiplying the angle be an integer:

$$\pm\frac{\sqrt{2IE}}{\hbar} = 0,\pm1,\pm2,\ldots$$

It is common to call this integer *m* and the write the solutions as

$$\psi(\rho,\phi) = NR(\rho)e^{im\phi} \;\; ; \quad m = 0,\pm1,\pm2,\ldots$$

The quantum number m is the orbital magnetic quantum number used to identify the eigenstates of L_z, which obey the eigenvalue equation $L_z|m\rangle = m\hbar|m\rangle$. We can now identify the energy eignevalues as

$$E_{|m|} = m^2\frac{\hbar^2}{2I}$$

These energy states are two-fold degenerate (except $m = 0$) because the energy is the same whether the particle rotates in a clockwise or a counterclockwise direction. Another way to see this is to note that the $|m\rangle$ eigenstates of L_z are also eigenstates of L_z^2:

$$L_z^2|m\rangle = m^2\hbar^2|m\rangle$$

but the $|m\rangle$ and $|-m\rangle$ states have the same L_z^2 eignevalue. Because $H = L_z^2/2I$, they must also have the same energy eignevalue.

12.5.2 The matrices for spin ½ are (can use *S* or *J* labels here)

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{S}^2 \doteq \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_+ \doteq \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- \doteq \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

For angular momentum 1, the matrices are

$$J_x \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_y \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_z \doteq \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{J}^2 \doteq 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad J_+ \doteq \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- \doteq \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For spin 3/2, the matrices are

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix} S_z \doteq \hbar\begin{pmatrix} +\frac{3}{2} & 0 & 0 & 0 \\ 0 & +\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

$$\mathbf{S}^2 \doteq \tfrac{15}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_+ \doteq \hbar\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad S_- \doteq \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

For $j = 1/2$, the commutator we want is

$$\begin{aligned}
[S_x,S_y] &\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&\doteq \left(\frac{\hbar}{2}\right)^2\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] \\
&\doteq \left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \doteq i\hbar\left(\frac{\hbar}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= i\hbar S_z
\end{aligned}$$

For $j = 1$, the commutator is

$$\begin{aligned}
[J_x,J_y] &\doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&\doteq \frac{\hbar^2}{2}\left[\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}\right] \\
&\doteq \frac{\hbar^2}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} \doteq i\hbar\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= i\hbar J_z
\end{aligned}$$

For $j = 3/2$, the commutator we want is

$$[S_x,S_y] \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix} -$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\doteq \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 3i & 0 & -i2\sqrt{3} & 0 \\ 0 & i & 0 & -i2\sqrt{3} \\ i2\sqrt{3} & 0 & -i & 0 \\ 0 & i2\sqrt{3} & 0 & -3i \end{pmatrix} - \begin{pmatrix} -3i & 0 & -i2\sqrt{3} & 0 \\ 0 & -i & 0 & -i2\sqrt{3} \\ i2\sqrt{3} & 0 & i & 0 \\ 0 & i2\sqrt{3} & 0 & 3i \end{pmatrix}\right]$$

$$\doteq \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 6i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & -6i \end{pmatrix} \doteq i\hbar\hbar \begin{pmatrix} +\frac{3}{2} & 0 & 0 & 0 \\ 0 & +\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

$$= i\hbar S_z$$

12.5.13 The spherical harmonics we want are

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

To write these in Cartesian coordinates, use

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

to get

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$
$$Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi \pm i\sin\phi) = \mp\sqrt{\frac{3}{4\pi}}\frac{(x\pm iy)}{\sqrt{2}r}$$

Invert these to get

$$z = \sqrt{\frac{4\pi}{3}} r Y_1^0(\theta,\phi)$$
$$x = \sqrt{\frac{4\pi}{3}} \sqrt{2} r \frac{1}{2}\left[Y_1^{-1}(\theta,\phi) - Y_1^1(\theta,\phi)\right]$$
$$y = \sqrt{\frac{4\pi}{3}} \sqrt{2} r \frac{1}{(-2i)}\left[Y_1^{-1}(\theta,\phi) + Y_1^1(\theta,\phi)\right]$$

Now rewrite the wave function using these expressions

$$\begin{aligned}
\psi &= N(x+y+2z)e^{-\alpha r} \\
&= N\sqrt{\frac{4\pi}{3}}\left\{\sqrt{2}r\frac{1}{2}\left[Y_1^{-1}(\theta,\phi) - Y_1^1(\theta,\phi)\right] + \sqrt{2}r\frac{1}{(-2i)}\left[Y_1^{-1}(\theta,\phi) + Y_1^1(\theta,\phi)\right] + 2rY_1^0(\theta,\phi)\right\}e^{-\alpha r} \\
&= N\sqrt{\frac{4\pi}{3}} r e^{-\alpha r}\left\{\frac{i-1}{\sqrt{2}}Y_1^1(\theta,\phi) + \frac{i+1}{\sqrt{2}}Y_1^{-1}(\theta,\phi) + 2Y_1^0(\theta,\phi)\right\}
\end{aligned}$$

To find the probability of measuring L_z, project the wave function onto the L_z eigenstate in question, square the amplitude, and then sum over all possible ways to obtain that probability. If the state was expanded in terms of discrete basis states $|n\ell m\rangle$, where the eigenvalues n refer to the other commuting observable (e.g. H), then we would express this as

$$\mathcal{P}_{L_z=mh} = \sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}\left|\langle n\ell m|\psi\rangle\right|^2$$

For the wave function above, we have written it in a way to make the $|\ell m\rangle$ aspect obvious, but we are using the continuous radial coordinate basis $|r\rangle$. In that case we must integrate over all possible values of that eigenvalue (see Eqn. 12.5.38):

$$\mathcal{P}_{L_z=mh} = \sum_{\ell=0}^{\infty}\int_0^{\infty}\left|\langle r\ell m|\psi\rangle\right|^2 r^2\,dr$$

Now rewrite the angular part of the wave function expression above in terms of the $|\ell m\rangle$ eigenstates, giving

$$\begin{aligned}
\mathcal{P}_{L_z=mh} &= \sum_{\ell=0}^{\infty}\int_0^{\infty}\left|\langle \ell m| N\sqrt{\frac{4\pi}{3}} r e^{-\alpha r}\left\{\frac{i-1}{\sqrt{2}}|11\rangle + \frac{i+1}{\sqrt{2}}|1,-1\rangle + 2|10\rangle\right\}\right|^2 r^2\,dr \\
&= |N|^2\frac{4\pi}{3}\sum_{\ell=0}^{\infty}\int_0^{\infty}\left|\langle \ell m|\left\{\frac{i-1}{\sqrt{2}}|11\rangle + \frac{i+1}{\sqrt{2}}|1,-1\rangle + 2|10\rangle\right\}\right|^2 r^2 e^{-2\alpha r} r^2\,dr \\
&= |N|^2\frac{4\pi}{3}\sum_{\ell=0}^{\infty}\left|\delta_{\ell 1}\left(\frac{i-1}{\sqrt{2}}\delta_{m1} + \frac{i+1}{\sqrt{2}}\delta_{m,-1} + 2\delta_{m0}\right)\right|^2\int_0^{\infty}e^{-2\alpha r}r^4\,dr
\end{aligned}$$

Note that (1) the square of a Kronecker delta is the same Kronecker delta because $0^2 = 0$ and $1^2 = 1$, (2) there are no cross terms in the square of a sum of Kronecker deltas because they are mutually exclusive, and (3) the Kronecker delta $\delta_{\ell 1}$ collapses the sum. Hence we get

$$\mathcal{P}_{L_z = mh} = |N|^2 \frac{4\pi}{3} \left\{ \delta_{m1} \left| \frac{i-1}{\sqrt{2}} \right|^2 + \delta_{m,-1} \left| \frac{i+1}{\sqrt{2}} \right|^2 + \delta_{m0} |2|^2 \right\} \int_0^\infty e^{-2\alpha r} r^4 \, dr$$

$$= |N|^2 \frac{4\pi}{3} \left\{ \delta_{m1} + \delta_{m,-1} + 4\delta_{m0} \right\} \int_0^\infty e^{-2\alpha r} r^4 \, dr$$

Thus the three probabilities are

$$\mathcal{P}_{L_z = +1h} = 1 \left\{ |N|^2 \frac{4\pi}{3} \int_0^\infty e^{-2\alpha r} r^4 \, dr \right\}$$

$$\mathcal{P}_{L_z = -1h} = 1 \left\{ |N|^2 \frac{4\pi}{3} \int_0^\infty e^{-2\alpha r} r^4 \, dr \right\}$$

$$\mathcal{P}_{L_z = 0h} = 4 \left\{ |N|^2 \frac{4\pi}{3} \int_0^\infty e^{-2\alpha r} r^4 \, dr \right\}$$

We could do the integral in the curly bracket and then find *N*, noting that the sum of these three probabilities must sum to unity. But why use up our precious brain cells? The terms in the curly brackets are identical, so we can normalize with an overall factor of 1/6 to get:

$$\mathcal{P}_{L_z = +1h} = \frac{1}{6}$$

$$\mathcal{P}_{L_z = -1h} = \frac{1}{6}$$

$$\mathcal{P}_{L_z = 0h} = \frac{4}{6} = \frac{2}{3}$$

2. $H = \varepsilon_\alpha a^+ a + \varepsilon_\beta b^+ b + g(a^+ b + b^+ a)$

$a^+ a + a a^+ = b^+ b + b b^+ = 1$

$aa = bb = 0$

$[a, b] = [a, b^+] = 0$

$N_\alpha = a^+ a, \quad N_\beta = b^+ b, \quad N = N_\alpha + N_\beta$

a) Find eigenvalues of N_α

Note that $N_\alpha N_\alpha = a^+ a a^+ a = a^+(1 - a^+ a)a$
$= a^+ a - a^+ a^+ a a$
$= a^+ a$ since $aa = 0$
$\Rightarrow N_\alpha N_\alpha = N_\alpha$

This is the unique feature!

Now consider some eigenvector of N_α, which we will label with the eigenvalue n_α. That is:

$N_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle$ is eigenvalue eqn.

Now apply N_α to both sides

$N_\alpha N_\alpha |n_\alpha\rangle = n_\alpha N_\alpha |n_\alpha\rangle$

$N_\alpha |n_\alpha\rangle = n_\alpha N_\alpha |n_\alpha\rangle$ since $N_\alpha N_\alpha = N_\alpha$

$\Rightarrow n_\alpha |n_\alpha\rangle = n_\alpha^2 |n_\alpha\rangle$

This is only satisfied (at least for non-null vectors $|n_\alpha\rangle$) if $n_\alpha = n_\alpha^2$

So this is the eqn which tells us what our eigenvalue spectrum is.

$$n_\alpha = n_\alpha^2$$

$$\Rightarrow n_\alpha (n_\alpha - 1) = 0$$

solns: $\boxed{n_\alpha = 0 \\ n_\alpha = 1}$

So there are only 2 possible eigenvalues: $\underline{\underline{0, 1}}$

You can arrive at this same result by following arguments for harmonic oscillator spectrum in book. Exception is that one cannot make a complete ladder of states since $aa = a^\dagger a^\dagger = 0$.

b) We want to show that $N = N_\alpha + N_\beta$ is a constant of the motion. To do this we must show that $[H, N] = 0$.

$$[H, N] = [\mathcal{E}_\alpha a^\dagger a + \mathcal{E}_\beta b^\dagger b + g(a^\dagger b + b^\dagger a), a^\dagger a + b^\dagger b]$$

but N_α commutes w/ itself, as does N_β

$$\Rightarrow [H, N] = g[a^\dagger b + b^\dagger a, a^\dagger a + b^\dagger b]$$

$$= g\{[a^\dagger b, a^\dagger a] + [a^\dagger b, b^\dagger b] + [b^\dagger a, a^\dagger a] + [b^\dagger a, b^\dagger b]\}$$

$$[a^\dagger b, a^\dagger a] = a^\dagger b a^\dagger a - a^\dagger a a^\dagger b$$
$$= a^\dagger a^\dagger b a - (1 - a a^\dagger) a^\dagger b \qquad \text{since } [a^\dagger, b] = 0 \text{ \& } a^\dagger a + a a^\dagger = 1$$
$$= 0 - a^\dagger b + 0$$

$$\Rightarrow [a^\dagger b, a^\dagger a] = -a^\dagger b$$

$$[a^\dagger b, b^\dagger b] = a^\dagger b b^\dagger b - b^\dagger b a^\dagger b$$
$$= a^\dagger b (1 - b b^\dagger)$$
$$= a^\dagger b$$

$$[b^\dagger a, a^\dagger a] = b^\dagger a a^\dagger a - a^\dagger a b^\dagger a$$
$$= b^\dagger a (1 - a a^\dagger)$$
$$= b^\dagger a$$

$$[b^+a, b^+b] = b^+ab^+b - b^+bb^+a$$
$$= -(1-bb^+)b^+a$$
$$= -b^+a$$

$$\Rightarrow [H,N] = g\{-a^+b + a^+b + b^+a - b^+a\}$$

$$\boxed{[H,N] = 0}$$

$\Rightarrow N$ is a constant of the motion.

c) N_α + N_β form a C.S.C.O.
Thus their simultaneous eigenfunctions form a complete basis for this problem.
Clearly N_β is just like N_α & so has eigenvalues 0, 1.
Thus there are only four possible tensor product states.
We will use this as our basis.

Basis $\{|n_\alpha\rangle|n_\beta\rangle\} \Leftrightarrow \{|n_\alpha n_\beta\rangle\}$

Four states are $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

In this basis H is clearly not diagonal, since the g terms cause coupling.
To write down the matrix (4x4) for H we must find out what the operators (a, b, a^+, b^+) do to the eigenstates.

They are just like Harmonic Oscillator operators:

For example

$$a|0\rangle = 0$$

In our case this is clear, since $aN_\alpha = aa^+a = a(1-aa^+) = a$

$$\Rightarrow aN_\alpha|0\rangle = a|0\rangle$$
$$\Rightarrow 0 = a|0\rangle$$

Now consider $N_\alpha a = a^\dagger a a = 0$

$\Rightarrow N_\alpha a|1\rangle = 0$

$\Rightarrow a|1\rangle$ is eigenvector of N_α w/ eigenvalue $n_\alpha = 0$

$\Rightarrow |0\rangle = c\, a|1\rangle$ where c is inserted so that $|0\rangle$, $|1\rangle$ remain normalized

$$\Rightarrow \langle 0|0\rangle = 1$$
$$= \langle 1|a^\dagger a|1\rangle |c|^2$$
$$= \langle 1|N_\alpha|1\rangle |c|^2$$
$$= 1 \cdot |c|^2 \cdot \langle 1|1\rangle$$

$\Rightarrow |c|^2 = 1$ choose as real & positive

$\Rightarrow a|1\rangle = |0\rangle$

Now use: $N_\alpha a^\dagger = a^\dagger a a^\dagger = a^\dagger(1 - a^\dagger a) = a^\dagger$
$a^\dagger N_\alpha = a^\dagger a^\dagger a = 0$

$\Rightarrow N_\alpha a^\dagger|0\rangle = a^\dagger|0\rangle$

So $a^\dagger|0\rangle$ is eigenvector of N_α w/ eigenvalue 1

$\Rightarrow a^\dagger|0\rangle = c|1\rangle$ again put in c to keep norm.

$\Rightarrow \langle 0|a a^\dagger|0\rangle = |c|^2$

$$\langle 0|(1 - a^\dagger a)|0\rangle = |c|^2$$
$$\langle 0|(1 - N_\alpha)|0\rangle = |c|^2$$
$$\Rightarrow |c| = 1$$ choose real & positive

$\Rightarrow a^\dagger|0\rangle = |1\rangle$

Next: $a^\dagger N_\alpha|1\rangle = 0$
$a^\dagger|1\rangle = 0$

Now we have four eqns:

$a|0\rangle = 0$
$a^+|0\rangle = |1\rangle$
$a|1\rangle = |0\rangle$
$a^+|1\rangle = 0$ ← only this one is different than H.O.

Now write these in terms of our 4 basis states

$a|0\, n_\beta\rangle = 0$ $\quad$ $b|n_\alpha\, 0\rangle = 0$
$a|1\, n_\beta\rangle = |0\, n_\beta\rangle$ $\quad$ $b|n_\alpha\, 1\rangle = |n_\alpha\, 0\rangle$
$a^+|0\, n_\beta\rangle = |1\, n_\beta\rangle$ $\quad$ $b^+|n_\alpha\, 0\rangle = |n_\alpha\, 1\rangle$
$a^+|1\, n_\beta\rangle = 0$ $\quad$ $b^+|n_\alpha\, 1\rangle = 0$

Now find matrix for H in this basis

$$H = \epsilon_\alpha a^+a + \epsilon_\beta b^+b + g(a^+b + b^+a)$$

$$\langle n_\alpha n_\beta|\epsilon_\alpha a^+a|n_\alpha' n_\beta'\rangle = \epsilon_\alpha \langle n_\alpha n_\beta|N_\alpha|n_\alpha' n_\beta'\rangle$$
$$= \epsilon_\alpha n_\alpha' \delta_{n_\alpha n_\alpha'} \delta_{n_\beta n_\beta'}$$

$$\langle n_\alpha n_\beta|\epsilon_\beta b^+b|n_\alpha' n_\beta'\rangle = \epsilon_\beta n_\beta' \delta_{n_\alpha n_\alpha'} \delta_{n_\beta n_\beta'}$$

$$\langle n_\alpha n_\beta|g a^+b|n_\alpha' n_\beta'\rangle = g\, \delta_{n_\alpha 1} \delta_{n_\alpha' 0} \delta_{n_\beta 0} \delta_{n_\beta' 1}$$

$$\langle n_\alpha n_\beta|g b^+a|n_\alpha' n_\beta'\rangle = g\, \delta_{n_\alpha 0} \delta_{n_\alpha' 1} \delta_{n_\beta 1} \delta_{n_\beta' 0}$$

$|n'_\alpha n'_\beta\rangle$:

$|00\rangle \quad |10\rangle \quad |01\rangle \quad |11\rangle$

$$\Rightarrow H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_\alpha & g & 0 \\ 0 & g & \epsilon_\beta & 0 \\ 0 & 0 & 0 & \epsilon_\alpha+\epsilon_\beta \end{pmatrix} \begin{matrix} |n_\alpha n_\beta\rangle \\ |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{matrix}$$

Now to find energy eigenvalues just diagonalize matrix. By inspection $E=0$, $\epsilon_\alpha+\epsilon_\beta$ will be two of the energy states, since those states are not mixed with other states.

That is: $H|00\rangle = 0$

$$H|11\rangle = (\epsilon_\alpha+\epsilon_\beta)|11\rangle$$

other 2 states will be mixtures of $|01\rangle$, $|10\rangle$

Diagonalize:

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & \epsilon_\alpha-\lambda & g & 0 \\ 0 & g & \epsilon_\beta-\lambda & 0 \\ 0 & 0 & 0 & \epsilon_\alpha+\epsilon_\beta-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)(\epsilon_\alpha+\epsilon_\beta-\lambda)\cdot\left[(\epsilon_\alpha-\lambda)(\epsilon_\beta-\lambda)-g^2\right]=0$$

$$\lambda\left[\lambda-(\epsilon_\alpha+\epsilon_\beta)\right]\left[\lambda^2-\lambda(\epsilon_\alpha+\epsilon_\beta)+\epsilon_\alpha\epsilon_\beta-g^2\right]=0$$

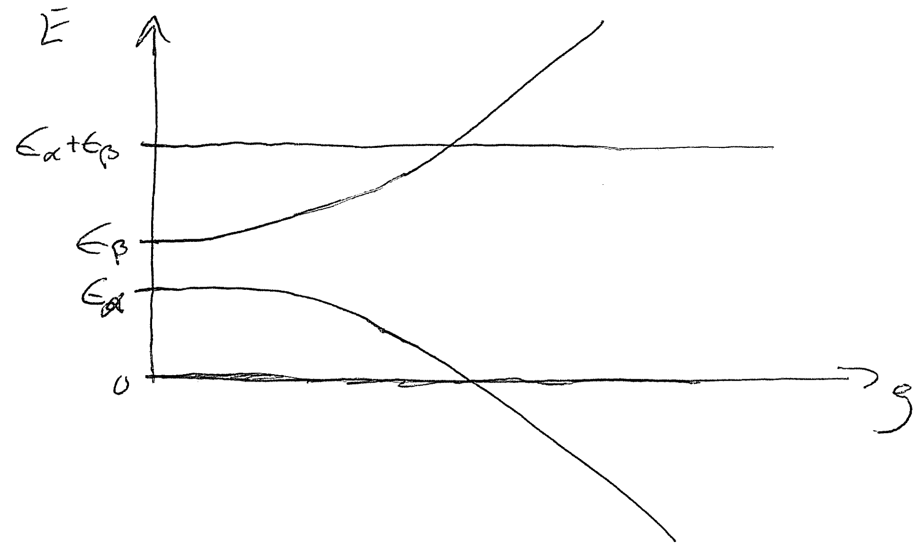
solns: $\lambda = 0$, $\epsilon_\alpha+\epsilon_\beta$ as expected

plus quadratic solns:

$$\lambda = \frac{\epsilon_\alpha + \epsilon_\beta}{2} \pm \sqrt{\left(\frac{\epsilon_\alpha + \epsilon_\beta}{2}\right)^2 - \epsilon_\alpha \epsilon_\beta + g^2}$$

$$\boxed{\lambda = \frac{\epsilon_\alpha + \epsilon_\beta}{2} \pm \sqrt{\left(\frac{\epsilon_\alpha - \epsilon_\beta}{2}\right)^2 + g^2}}$$

Two other energy eigenvalues.
So as a function of g energies look like,



g only couples 2 states.