

1.6.2 Ω and Λ are Hermitian ($\Omega = \Omega^\dagger$ and $\Lambda = \Lambda^\dagger$).

(1) Consider the adjoint of the product:

$$\begin{aligned}(\Omega\Lambda)^\dagger &= \Lambda^\dagger\Omega^\dagger \\ &= \Lambda\Omega\end{aligned}$$

(2) For the sum of the products, we get

$$\begin{aligned}(\Omega\Lambda + \Lambda\Omega)^\dagger &= (\Omega\Lambda)^\dagger + (\Lambda\Omega)^\dagger \\ &= \Lambda^\dagger\Omega^\dagger + \Omega^\dagger\Lambda^\dagger \\ &= \Lambda\Omega + \Omega\Lambda \\ &= \Omega\Lambda + \Lambda\Omega\end{aligned}$$

Hence, the sum of the products of two Hermitian operators is itself Hermitian.

(3) For the commutator, we get

$$\begin{aligned}[\Omega, \Lambda] &= \Omega\Lambda - \Lambda\Omega \\ [\Omega, \Lambda]^\dagger &= (\Omega\Lambda - \Lambda\Omega)^\dagger \\ &= (\Omega\Lambda)^\dagger - (\Lambda\Omega)^\dagger \\ &= \Lambda^\dagger\Omega^\dagger - \Omega^\dagger\Lambda^\dagger \\ &= \Lambda\Omega - \Omega\Lambda \\ &= -(\Omega\Lambda - \Lambda\Omega)\end{aligned}$$

Hence, the commutator of two Hermitian operators is anti-Hermitian ($A = -A^\dagger$).

(4) If we multiply the commutator by i , then we get

$$\begin{aligned}[\Omega, \Lambda] &= \Omega\Lambda - \Lambda\Omega \\ (i[\Omega, \Lambda])^\dagger &= (i)^\dagger [\Omega, \Lambda]^\dagger \\ &= -i(-[\Omega, \Lambda]) \\ &= i[\Omega, \Lambda]\end{aligned}$$

which implies that $i[\Omega, \Lambda]$ is Hermitian.

9.4.3 For a one-dimensional hydrogen atom, the Hamiltonian is

$$H = \frac{P^2}{2m} - \frac{e^2}{R}$$

The expectation value of the energy is

$$\langle H \rangle = \frac{\langle P^2 \rangle}{2m} - \left\langle \frac{e^2}{R} \right\rangle \approx \frac{\langle P^2 \rangle}{2m} - \frac{e^2}{\langle (R^2) \rangle^{1/2}}$$

Assume that $\langle P \rangle = 0$ and $\langle R \rangle = 0$ and use $\langle \Omega^2 \rangle = (\Delta\Omega)^2 + \langle \Omega \rangle^2$ to get

$$\langle H \rangle \approx \frac{(\Delta P)^2}{2m} - \frac{e^2}{\Delta R}$$

Assuming the uncertainty relation $\Delta P \Delta R \geq \hbar/2$, we then have

$$\langle H \rangle \geq \frac{\hbar^2}{8m(\Delta R)^2} - \frac{e^2}{\Delta R}$$

Minimizing this with respect to ΔR , gives

$$\frac{d\langle H \rangle}{d(\Delta R)} = \frac{-2\hbar^2}{8m(\Delta R)^3} + \frac{e^2}{(\Delta R)^2} = 0 \quad \Rightarrow \quad (\Delta R) = \frac{\hbar^2}{4me^2} = \frac{a_0}{4}$$

The resultant energy is

$$\langle H \rangle \geq \frac{\hbar^2}{8m(\hbar^2/4me^2)^2} - \frac{e^2}{(\hbar^2/4me^2)} = -\frac{2me^4}{\hbar^2} = -2\alpha^2 mc^2,$$

which is 4 times larger than the actual energy.