Problem 9.17

Equation 9.106 $\Rightarrow \beta = 2.42$; Eq. 9.110 $\Rightarrow$

$$\alpha = \frac{\sqrt{1-(\sin\theta/2.42)^2}}{\cos\theta}.$$

(a) $\theta = 0 \Rightarrow \alpha = 1$. Eq. 9.109 $\Rightarrow \left(\frac{E_{0_R}}{E_{0_I}}\right) = \frac{\alpha-\beta}{\alpha+\beta} = \frac{1-2.42}{1+2.42} = -\frac{1.42}{3.42} = \boxed{-0.415;}$

$\left(\frac{E_{0_T}}{E_{0_I}}\right) = \frac{2}{\alpha+\beta} = \frac{2}{3.42} = \boxed{0.585.}$

(b) Equation 9.112 $\Rightarrow \theta_B = \tan^{-1}(2.42) = \boxed{67.5^\circ.}$

(c) $E_{0_R} = E_{0_T} \Rightarrow \alpha - \beta = 2; \alpha = \beta + 2 = 4.42;$

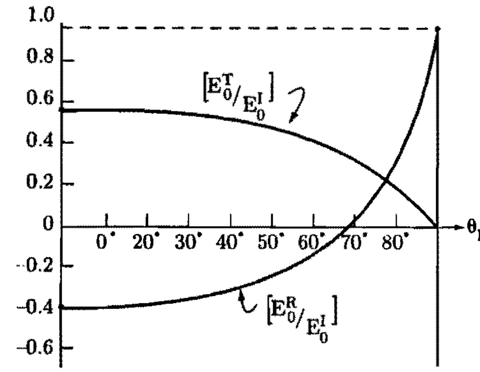
$(4.42)^2\cos^2\theta = 1 - \sin^2\theta/(2.42)^2;$

$(4.42)^2(1-\sin^2\theta) = (4.42)^2 - (4.42)^2\sin^2\theta$

$= 1 - 0.171\sin^2\theta;\ 19.5 - 1 = (19.5 - 0.17)\sin^2\theta;$

$18.5 = 19.3\sin^2\theta;\ \sin^2\theta = 18.5/19.3 = 0.959;$

$\sin\theta = 0.979;\ \boxed{\theta = 78.3^\circ.}$

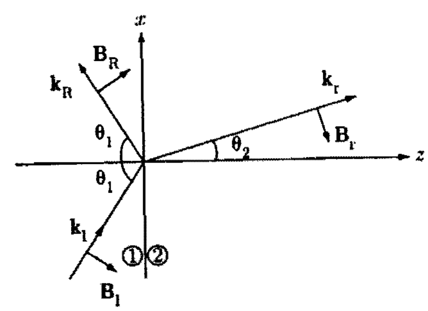


Problem 9.16

$$\left\{\begin{array}{lcl} \tilde{\mathbf{E}}_I &=& \tilde{E}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_I &=& \frac{1}{v_1}\tilde{E}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}(-\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{array}\right\}$$

$$\left\{\begin{array}{lcl} \tilde{\mathbf{E}}_R &=& \tilde{E}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_R &=& \frac{1}{v_1}\tilde{E}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}(\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{array}\right\}$$

$$\left\{\begin{array}{lcl} \tilde{\mathbf{E}}_T &=& \tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_T &=& \frac{1}{v_2}\tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_2\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{array}\right\}$$



Boundary conditions: $\left\{\begin{array}{ll} \text{(i) } \epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp, & \text{(iii) } \mathbf{E}_1^\parallel = \mathbf{E}_2^\parallel, \\ \text{(ii) } B_1^\perp = B_2^\perp, & \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_1^\parallel = \frac{1}{\mu_2}\mathbf{B}_2^\parallel. \end{array}\right.$

Law of refraction: $\frac{\sin\theta_2}{\sin\theta_1} = \frac{v_2}{v_1}$. [Note: $\mathbf{k}_I\cdot\mathbf{r} - \omega t = \mathbf{k}_R\cdot\mathbf{r} - \omega t = \mathbf{k}_T\cdot\mathbf{r} - \omega t$, at $z = 0$, so we can drop all exponential factors in applying the boundary conditions.]

Boundary condition (i): $0 = 0$ (trivial). Boundary condition (iii): $\boxed{\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}.}$

Boundary condition (ii): $\frac{1}{v_1}\tilde{E}_{0_I}\sin\theta_1 + \frac{1}{v_1}\tilde{E}_{0_R}\sin\theta_1 = \frac{1}{v_2}\tilde{E}_{0_T}\sin\theta_2 \Rightarrow \tilde{E}_{0_I} + \tilde{E}_{0_R} = \left(\frac{v_1\sin\theta_2}{v_2\sin\theta_1}\right)\tilde{E}_{0_T}.$
But the term in parentheses is 1, by the law of refraction, so this is the same as (ii).

Boundary condition (iv): $\frac{1}{\mu_1}\left[\frac{1}{v_1}\tilde{E}_{0_I}(-\cos\theta_1) + \frac{1}{v_1}\tilde{E}_{0_R}\cos\theta_1\right] = \frac{1}{\mu_2 v_2}\tilde{E}_{0_T}(-\cos\theta_2) \Rightarrow$

$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \left(\frac{\mu_1 v_1\cos\theta_2}{\mu_2 v_2\cos\theta_1}\right)\tilde{E}_{0_T}$. Let $\boxed{\alpha \equiv \frac{\cos\theta_2}{\cos\theta_1};\ \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}.}$ Then $\boxed{\tilde{E}_{0_I} - \tilde{E}_{0_R} = \alpha\beta\tilde{E}_{0_T}.}$

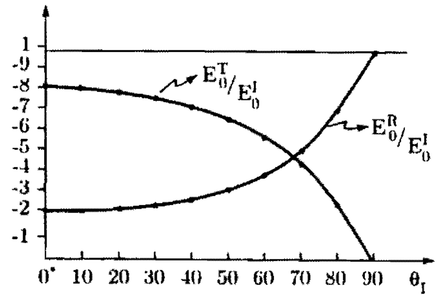
Solving for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$: $2\tilde{E}_{0_I} = (1+\alpha\beta)\tilde{E}_{0_T} \Rightarrow \tilde{E}_{0_T} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{0_I};$

$\tilde{E}_{0_R} = \tilde{E}_{0_T} - \tilde{E}_{0_I} = \left(\frac{2}{1+\alpha\beta} - \frac{1+\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0_I} \Rightarrow \tilde{E}_{0_R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0_I}.$

Since α and β are positive, it follows that $2/(1+\alpha\beta)$ is positive, and hence the *transmitted* wave is *in phase* with the incident wave, and the (real) amplitudes are related by $\boxed{E_{0_T} = \left(\frac{2}{1+\alpha\beta}\right)E_{0_I}.}$ The *reflected* wave is in phase if $\alpha\beta < 1$ and 180° out of phase if $\alpha\beta < 1$; the (real) amplitudes are related by $\boxed{E_{0_R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0_I}.}$

These are the **Fresnel equations** for polarization perpendicular to the plane of incidence.

To construct the graphs, note that $\alpha\beta = \beta\frac{\sqrt{1-\sin^2\theta/\beta^2}}{\cos\theta} = \frac{\sqrt{\beta^2-\sin^2\theta}}{\cos\theta}$, where θ is the angle of incidence, so, for $\beta = 1.5$, $\alpha\beta = \frac{\sqrt{2.25-\sin^2\theta}}{\cos\theta}$.



Is there a Brewster's angle? Well, $E_{0_R} = 0$ would mean that $\alpha\beta = 1$, and hence that

$$\alpha = \frac{\sqrt{1-(v_2/v_1)^2\sin^2\theta}}{\cos\theta} = \frac{1}{\beta} = \frac{\mu_2 v_2}{\mu_1 v_1}, \text{ or } 1 - \left(\frac{v_2}{v_1}\right)^2 \sin^2\theta = \left(\frac{\mu_2 v_2}{\mu_1 v_1}\right)^2 \cos^2\theta, \text{ so}$$

$1 = \left(\frac{v_2}{v_1}\right)^2\left[\sin^2\theta + (\mu_2/\mu_1)^2\cos^2\theta\right]$. Since $\mu_1 \approx \mu_2$, this means $1 \approx (v_2/v_1)^2$, which is only true for optically indistinguishable media, in which case there is of *course* no reflection—but that would be true at *any* angle, not just at a special "Brewster's angle". [If μ_2 were substantially different from μ_1, and the relative velocities were just right, it *would* be possible to get a Brewster's angle for this case, at

$$\left(\frac{v_1}{v_2}\right)^2 = 1 - \cos^2\theta + \left(\frac{\mu_2}{\mu_1}\right)^2\cos^2\theta \Rightarrow \cos^2\theta = \frac{(v_1/v_2)^2 - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\mu_2\epsilon_2/\mu_1\epsilon_1) - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\epsilon_2/\epsilon_1) - (\mu_1/\mu_2)}{(\mu_2/\mu_1) - (\mu_1/\mu_2)}.$$

But the media would be very peculiar.]

By the same token, δ_R is either always 0, or always π, for a given interface—it does not switch over as you change θ, the way it does for polarization in the plane of incidence. In particular, if $\beta = 3/2$, then $\alpha\beta > 1$, for

$$\alpha\beta = \frac{\sqrt{2.25-\sin^2\theta}}{\cos\theta} > 1 \text{ if } 2.25 - \sin^2\theta > \cos^2\theta, \text{ or } 2.25 > \sin^2\theta + \cos^2\theta = 1. \checkmark$$

In general, for $\beta > 1$, $\alpha\beta > 1$, and hence $\delta_R = \pi$. For $\beta < 1$, $\alpha\beta < 1$, and $\delta_R = 0$.

At normal incidence, $\alpha = 1$, so Fresnel's equations reduce to $E_{0_T} = \left(\frac{2}{1+\beta}\right)E_{0_I}$; $E_{0_R} = \left|\frac{1-\beta}{1+\beta}\right| E_{0_I}$, consistent with Eq. 9.82.

Reflection and Transmission coefficients: $\boxed{R = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2.}$ Referring to Eq. 9.116,

$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \boxed{\alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2.}$$

$$R + T = \frac{(1-\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{1 - 2\alpha\beta + \alpha^2\beta^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1. \checkmark$$