Problem 7.42

(a) Faraday's law says $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, so $\mathbf{E}=0 \Rightarrow \frac{\partial\mathbf{B}}{\partial t}=0 \Rightarrow \mathbf{B}(\mathbf{r})$ is independent of t.

(b) Faraday's law in integral form (Eq. 7.18) says $\oint \mathbf{E}\cdot d\mathbf{l} = -d\Phi/dt$. In the wire itself $\mathbf{E}=0$, so Φ through the loop is constant.

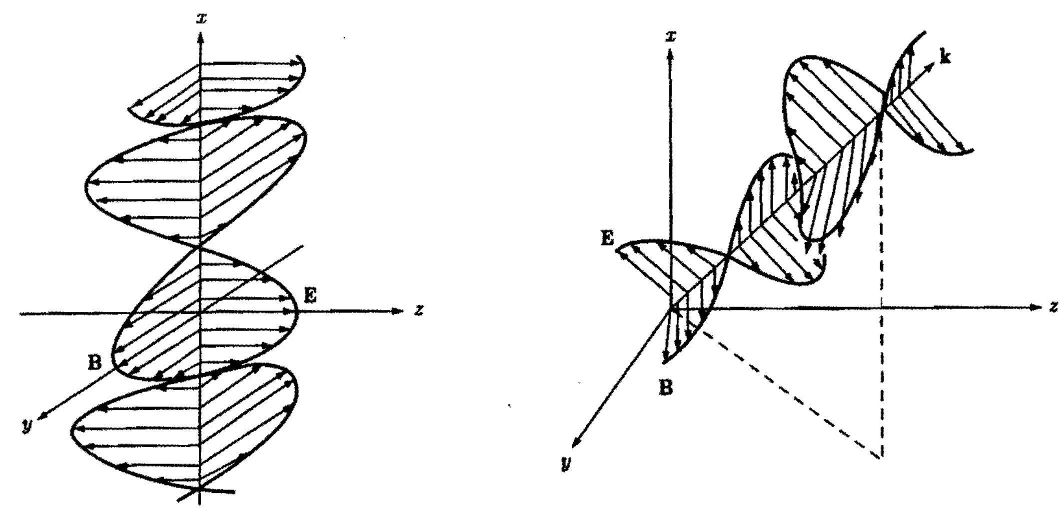
(c) Ampère-Maxwell$\Rightarrow$ $\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$, so $\mathbf{E}=0$, $\mathbf{B}=0 \Rightarrow \mathbf{J}=0$, and hence any current must be at the surface.

(d) From Eq. 5.68, a rotating shell produces a uniform magnetic field (inside): $\mathbf{B}=\frac{2}{3}\mu_0\sigma\omega a\hat{\mathbf{z}}$. So to *cancel* such a field, we need $\sigma\omega a = -\frac{3}{2}\frac{B_0}{\mu_0}$. Now $\mathbf{K}=\sigma\mathbf{v}=\sigma\omega a\sin\theta\,\hat{\boldsymbol{\phi}}$, so $\boxed{\mathbf{K} = -\frac{3B_0}{2\mu_0}\sin\theta\,\hat{\boldsymbol{\phi}}.}$

Problem 9.9

(a) $\boxed{\mathbf{k} = -\frac{\omega}{c}\hat{\mathbf{x}};\ \hat{\mathbf{n}}=\hat{\mathbf{z}}.}$ $\mathbf{k}\cdot\mathbf{r} = \left(-\frac{\omega}{c}\hat{\mathbf{x}}\right)\cdot(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}) = -\frac{\omega}{c}x$; $\mathbf{k}\times\hat{\mathbf{n}} = -\hat{\mathbf{x}}\times\hat{\mathbf{z}}=\hat{\mathbf{y}}$.

$$\boxed{\mathbf{E}(x,t) = E_0\cos\left(\frac{\omega}{c}x+\omega t\right)\hat{\mathbf{z}};\quad \mathbf{B}(x,t) = \frac{E_0}{c}\cos\left(\frac{\omega}{c}x+\omega t\right)\hat{\mathbf{y}}.}$$



(b) $\boxed{\mathbf{k} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}}{\sqrt{3}}\right);\ \hat{\mathbf{n}} = \frac{\hat{\mathbf{x}}-\hat{\mathbf{z}}}{\sqrt{2}}.}$ (Since $\hat{\mathbf{n}}$ is parallel to the xz plane, it must have the form $\alpha\,\hat{\mathbf{x}}+\beta\,\hat{\mathbf{z}}$; since $\hat{\mathbf{n}}\cdot\mathbf{k}=0, \beta=-\alpha$; and since it is a unit vector, $\alpha = 1/\sqrt{2}$.)

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega}{\sqrt{3}c}(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}) = \frac{\omega}{\sqrt{3}c}(x+y+z);\ \hat{\mathbf{k}}\times\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 1 & 1 & 1\\ 1 & 0 & -1\end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{\mathbf{x}}+2\,\hat{\mathbf{y}}-\hat{\mathbf{z}}).$$

$$\boxed{\begin{aligned}\mathbf{E}(x,y,z,t) &= E_0\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z)-\omega t\right]\left(\frac{\hat{\mathbf{x}}-\hat{\mathbf{z}}}{\sqrt{2}}\right);\\ \mathbf{B}(x,y,z,t) &= \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z)-\omega t\right]\left(\frac{-\hat{\mathbf{x}}+2\hat{\mathbf{y}}-\hat{\mathbf{z}}}{\sqrt{6}}\right).\end{aligned}}$$

Problem 9.10

$P = \frac{I}{c} = \frac{1.3\times10^3}{3.0\times10^8} = \boxed{4.3\times10^{-6}\,\mathrm{N/m^2}.}$ For a perfect reflector the pressure is twice as great: $\boxed{8.6\times10^{-6}\,\mathrm{N/m^2}.}$ Atmospheric pressure is $1.03\times10^5\,\mathrm{N/m^2}$, so the pressure of light on a reflector is $(8.6\times10^{-6})/(1.03\times10^5) = \boxed{8.3\times10^{-11}\text{ atmospheres.}}$

Problem 9.33

(a) (i) *Gauss's law:* $\nabla\cdot\mathbf{E} = \dfrac{1}{r\sin\theta}\dfrac{\partial E_\phi}{\partial\phi} = 0.$ ✓

(ii) *Faraday's law:*

$$\begin{aligned}
-\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\,\hat{\boldsymbol{\theta}} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[E_0\frac{\sin^2\theta}{r}\left(\cos u - \frac{1}{kr}\sin u\right)\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left[E_0\sin\theta\left(\cos u - \frac{1}{kr}\sin u\right)\right]\hat{\boldsymbol{\theta}}.
\end{aligned}$$

But $\dfrac{\partial}{\partial r}\cos u = -k\sin u;\ \dfrac{\partial}{\partial r}\sin u = k\cos u.$

$$= \frac{1}{r\sin\theta}\frac{E_0}{r}2\sin\theta\cos\theta\left(\cos u - \frac{1}{kr}\sin u\right)\hat{\mathbf{r}} - \frac{1}{r}E_0\sin\theta\left(-k\sin u + \frac{1}{kr^2}\sin u - \frac{1}{r}\cos u\right)\hat{\boldsymbol{\theta}}.$$

Integrating with respect to t, and noting that $\int\cos u\,dt = -\frac{1}{\omega}\sin u$ and $\int\sin u\,dt = \frac{1}{\omega}\cos u$, we obtain

$$\boxed{\mathbf{B} = \frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\mathbf{r}} + \frac{E_0\sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\hat{\boldsymbol{\theta}}.}$$

(iii) *Divergence of* **B**:

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{2E_0\cos\theta}{\omega}\left(\sin u + \frac{1}{kr}\cos u\right)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{E_0\sin^2\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right] \\
&= \frac{1}{r^2}\frac{2E_0\cos\theta}{\omega}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{2E_0\sin\theta\cos\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)
\end{aligned}$$

$$= \frac{2E_0\cos\theta}{\omega r^2}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u - k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right) = 0. \checkmark$$

(iv) *Ampére/Maxwell:*

$$\begin{aligned}
\nabla\times\mathbf{B} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\phi} \\
&= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[\frac{E_0\sin\theta}{\omega}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right] - \frac{\partial}{\partial\theta}\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\right]\right\}\hat{\phi} \\
&= \frac{E_0\sin\theta}{\omega r}\left(k^2\sin u - \frac{2}{kr^3}\cos u - \frac{1}{r^2}\sin u - \frac{1}{r^2}\sin u + \frac{k}{r}\cos u + \frac{2}{r^2}\sin u + \frac{2}{kr^3}\cos u\right)\hat{\phi} \\
&= \frac{k}{\omega}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi}. \\
\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \frac{1}{c^2}\frac{E_0\sin\theta}{r}\left(\omega\sin u + \frac{\omega}{kr}\cos u\right)\hat{\phi} = \frac{1}{c^2}\frac{\omega}{k}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} \\
&= \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \nabla\times\mathbf{B}. \checkmark
\end{aligned}$$

(b) *Poynting Vector:*

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{E_0\sin\theta}{\mu_0 r}\left(\cos u - \frac{1}{kr}\sin u\right)\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\theta}\right. \\
&\quad \left. + \frac{E_0\sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)(-\hat{\mathbf{r}})\right] \\
&= \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u) - \frac{1}{k^2r^2}\sin u\cos u\right]\hat{\theta}\right. \\
&\quad \left. - \sin\theta\left(-k\cos^2 u + \frac{1}{kr^2}\cos^2 u + \frac{1}{r}\sin u\cos u + \frac{1}{r}\sin u\cos u - \frac{1}{k^2r^3}\sin u\cos u - \frac{1}{kr^2}\sin^2 u\right)\hat{\mathbf{r}}\right\} \\
&= \boxed{\begin{aligned}&\frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\left(1 - \frac{1}{k^2r^2}\right)\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u)\right]\hat{\theta}\right. \\ &\quad \left. + \sin\theta\left[\left(-\frac{2}{r} + \frac{1}{k^2r^3}\right)\sin u\cos u + k\cos^2 u + \frac{1}{kr^2}(\sin^2 u - \cos^2 u)\right]\hat{\mathbf{r}}\right\}.\end{aligned}}
\end{aligned}$$

Averaging over a full cycle, using $\langle\sin u\cos u\rangle = 0$, $\langle\sin^2 u\rangle = \langle\cos^2 u\rangle = \frac{1}{2}$, we get the intensity:

$$\mathbf{I} = \langle\mathbf{S}\rangle = \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left(\frac{k}{2}\sin\theta\right)\hat{\mathbf{r}} = \boxed{\frac{E_0^2\sin^2\theta}{2\mu_0 c r^2}\hat{\mathbf{r}}.}$$

It points in the $\hat{\mathbf{r}}$ direction, and falls off as $1/r^2$, as we would expect for a spherical wave.

(c) $P = \int \mathbf{I}\cdot d\mathbf{a} = \frac{E_0^2}{2\mu_0 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi = \frac{E_0^2}{2\mu_0 c}2\pi\int_0^\pi \sin^3\theta\,d\theta = \boxed{\frac{4\pi}{3}\frac{E_0^2}{\mu_0 c}.}$