

Problem 3.3

Laplace's equation in *spherical* coordinates, for V dependent only on r , reads:

$$\nabla^2 V = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right) = 0 \Rightarrow r^2 \frac{dV}{dr} = c \text{ (constant)} \Rightarrow \frac{dV}{dr} = \frac{c}{r^2} \Rightarrow \boxed{V = -\frac{c}{r} + k.}$$

Example: potential of a uniformly charged sphere.

In *cylindrical* coordinates: $\nabla^2 V = \frac{1}{s} \frac{d}{ds} \left(s \frac{dV}{ds} \right) = 0 \Rightarrow s \frac{dV}{ds} = c \Rightarrow \frac{dV}{ds} = \frac{c}{s} \Rightarrow \boxed{V = c \ln s + k.}$

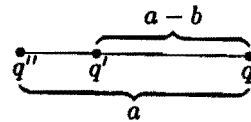
Example: potential of a long wire.

Problem 3.8

Place a second image charge, q'' , at the *center* of the sphere; this will not alter the fact that the sphere is an *equipotential*, but merely *increase* that potential from zero to $V_0 = \frac{1}{4\pi\epsilon_0} \frac{q''}{R}$;

$$\boxed{q'' = 4\pi\epsilon_0 V_0 R \text{ at center of sphere.}}$$

For a *neutral* sphere, $q' + q'' = 0$.



$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2} \right) = \frac{qq'}{4\pi\epsilon_0} \left(-\frac{1}{a^2} + \frac{1}{(a-b)^2} \right) \\ &= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2} = \frac{q(-Rq/a)(R^2/a)(2a-R^2/a)}{4\pi\epsilon_0 a^2(a-R^2/a)^2} \\ &= - \frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a} \right)^3 \frac{(2a^2 - R^2)}{(a^2 - R^2)^2}. \end{aligned}$$

(Drop the minus sign, because the problem asks for the force of *attraction*.)

Problem 3.15

Same format as Ex. 3.5, only the boundary conditions are:

$$\left\{ \begin{array}{l} \text{(i)} \quad V = 0 \quad \text{when } x = 0, \\ \text{(ii)} \quad V = 0 \quad \text{when } x = a, \\ \text{(iii)} \quad V = 0 \quad \text{when } y = 0, \\ \text{(iv)} \quad V = 0 \quad \text{when } y = a, \\ \text{(v)} \quad V = 0 \quad \text{when } z = 0, \\ \text{(vi)} \quad V = V_0 \quad \text{when } z = a. \end{array} \right.$$

This time we want sinusoidal functions in x and y , exponential in z :

$$X(x) = A \sin(kx) + B \cos(kx), \quad Y(y) = C \sin(l y) + D \cos(l y), \quad Z(z) = E e^{\sqrt{k^2+l^2}z} + G e^{-\sqrt{k^2+l^2}z}.$$

(i) $\Rightarrow B = 0$; (ii) $\Rightarrow k = n\pi/a$; (iii) $\Rightarrow D = 0$; (iv) $\Rightarrow l = m\pi/a$; (v) $\Rightarrow E + G = 0$. Therefore

$$Z(z) = 2E \sinh(\pi \sqrt{n^2 + m^2} z/a).$$

Putting this all together, and combining the constants, we have:

$$V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sin(n\pi x/a) \sin(m\pi y/a) \sinh(\pi \sqrt{n^2 + m^2} z/a).$$

It remains to evaluate the constants $C_{n,m}$, by imposing boundary condition (vi):

$$V_0 = \sum \sum [C_{n,m} \sinh(\pi \sqrt{n^2 + m^2} z/a)] \sin(n\pi x/a) \sin(m\pi y/a).$$

According to Eqs. 3.50 and 3.51:

$$C_{n,m} \sinh(\pi \sqrt{n^2 + m^2} z/a) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a \int_0^a \sin(n\pi x/a) \sin(m\pi y/a) dx dy = \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \frac{16V_0}{\pi^2 nm}, & \text{if both are odd.} \end{cases}$$

Therefore

$$V(x, y, z) = \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\dots} \sum_{m=1,3,5,\dots} \frac{1}{nm} \sin(n\pi x/a) \sin(m\pi y/a) \frac{\sinh(\pi \sqrt{n^2 + m^2} z/a)}{\sinh(\pi \sqrt{n^2 + m^2} z/a)}.$$

Problem 3.24

Picking $V = 0$ on the yz plane, with E_0 in the x direction, we have (Eq. 3.74):

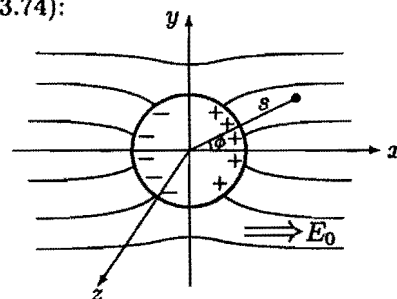
$$\left\{ \begin{array}{l} \text{(i)} \quad V = 0, \quad \text{when } s = R, \\ \text{(ii)} \quad V \rightarrow -E_0 x = -E_0 s \cos \phi, \quad \text{for } s \gg R. \end{array} \right.$$

Evidently $a_0 = b_0 = b_k = d_k = 0$, and $a_k = c_k = 0$ except for $k = 1$:

$$V(s, \phi) = \left(a_1 s + \frac{c_1}{s}\right) \cos \phi.$$

(i) $\Rightarrow c_1 = -a_1 R^2$; (ii) $\rightarrow a_1 = -E_0$. Therefore

$$V(s, \phi) = \left(-E_0 s + \frac{E_0 R^2}{s}\right) \cos \phi, \quad \text{or} \quad V(s, \phi) = -E_0 s \left[\left(\frac{R}{s}\right)^2 - 1\right] \cos \phi.$$



$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial s} \right|_{s=R} = -\epsilon_0 E_0 \left(-\frac{R^2}{s^2} - 1\right) \cos \phi \Big|_{s=R} = \boxed{2\epsilon_0 E_0 \cos \phi}.$$